Journal of Statistical Physics, Vol. 92, Nos. 3/4, 1998

Construction of the Hierarchical ϕ^4-Trajectory

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Received July 7, 1997; final May 1, 1998

We study the invariant unstable manifold of the trivial renormalization-group fixed point tangent to the ϕ^4-vertex in the hierarchical approximation. We parametrize it by a running ϕ^4-coupling with linear step β-function. The manifold is studied as a fixed point of the renormalization group composed with a flow of the running coupling. We present a rigorous construction of it beyond perturbation theory by means of a contraction mapping. Starting from a perturbative approximant of order seven, we obtain a convergent representation in dimensions $2 < D < 28/9$ with certain restrictions. The perturbative approximant is logarithmically divergent in $D = 3$ dimensions.

KEY WORDS: Hierarchical renormalization group; renormalized ϕ^4-trajectory; contraction mapping.

1. INTRODUCTION

The non-perturbative renormalization of the scalar ϕ^4-vertex is a central problem in Euclidean quantum field theory [GJ73, FO76, Gal78, Gal79, BCG+80, Bal83, GK85a, GK85b, GN85, FMRS87, P90, BDH93]. The key to its solution is the renormalization group [Wil71, Wil72, WK74, Gal85, GK83, GK86, GJ87, R91, Bry92, FFS92, BG95]. Its renormalization is a mapping through an increasing number of renormalization group transformations, simultaneously tuning the ϕ^4-coupling besides other parameters. The renormalization group brings an understanding of this process as a dynamical system on a space of effective interactions. The objects of principal interest in this dynamical system are the attractors, given by fixed points and their invariant manifolds. The invariant ϕ^4-curve of the trivial fixed point is a canonical example. We present its non-perturbative construction in the hierarchical renormalization group [D69, BS73,

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0022-4715/98/0800-0377$15.00/0

BS75, Ble77, CE77, Gal78, Gal79, GK82]. Our method builds upon [KW86a, KW86b, KW88a, KW88b, KW91, KW94], [P90, P93, Alb91], and [RW96]. The outcome is a non-perturbative renormalized ϕ^4-theory without a detour to a limit procedure [Wie97].

The hierarchical model is defined by a non-linear transformation R, acting on a space of functions $Z(\phi)$ of a complex variable ϕ. They represent local interaction Boltzmann weights. The transformation R has a trivial fixed point $Z_{UV}(\phi) = 1$. This fixed point represents the hierarchical massless scalar field. The derivative of R at this fixed point $Z_{UV}(\phi)$ is a linear operator, whose eigenfunctions are normal ordered products. One of these eigenfunctions is the normal ordered ϕ^4-vertex, i.e., the rescaled Hermite polynomial $:\phi^4:_v$. Its eigenvalue is L^{4-D}, where L is a scale parameter, and where D is a dimension parameter. We take both parameters to be real valued, with $L>1$ and $2<D<4$. Associated with the pair, given by $Z_{UV}(\phi)$ and $:\phi^4:_v$, is a curve $Z(\phi, g)$, parametrized by g, with the following properties:

(I) $Z(\phi, g)$ emerges from $Z_{UV}(\phi)$ tangent to $:\phi^4:_v$:

$$Z(\phi, g) = \mathrm{e}^{-g\,:\phi^4:_v}(1 + O(g^2)) \tag{1}$$

(II) $Z(\phi, g)$ is invariant under R up to an inverse flow of g:

$$R(Z)(\psi, g) = Z(\psi, \delta^{-1}(g)), \qquad \delta^{-1}(g) = L^{4-D}g + O(g^2) \tag{2}$$

This curve is called the ϕ^4-trajectory, and is an invariant curve in the unstable manifold of $Z_{UV}(\phi)$. Eqs. (1) and (2) still leave room for reparametrizations of g. A normal form in $D<4$ dimensions is given by:

(III) The flow function $\delta^{-1}(g)$ is linear:

$$\delta^{-1}(g) = L^{4-D}g \tag{3}$$

These properties (I), (II), and (III) determine $Z(\phi, g)$ uniquely as formal power series in g, except at certain special dimensions, where a resonance of power counting factors occur [RW96]. Formal perturbation theory is unfortunately divergent. We replace it by the following non-perturbative construction.

The properties (II) and (III) require $Z(\phi, g)$ to be a fixed point of $R \times \delta^{\star}$, the renormalization group composed with a flow of g. This property will be the starting point of our construction. We thus look for a non-trivial fixed point of $R \times \delta^{\star}$. The property (I) needs to be modified. The reason is that the $O(g^2)$ corrections cannot be uniform in ϕ. Instead we split $Z(\phi, g)$ into two pieces, an approximate fixed point $Z_1(\phi, g)$, which we compute from perturbation theory, and a correction term $Z_2(\phi, g)$. We choose both terms such that $Z(\phi, g)$ is element of an invariant cone in a certain Banach space of functions. We determine $Z_1(\phi, g)$ sufficiently accurately such that the transformation of $Z_2(\phi, g)$ becomes a contraction mapping. It follows that there exists a unique fixed point in our invariant cone. Furthermore, the iteration of the contraction mapping yields a convergent representation. It is this representation which replaces the perturbative iteration of (I), (II), and (III).

The contraction mapping method was suggested to us by the previous construction of the hierachical non-trivial fixed point and its unstable manifold [KW86a, KW86b, KW88a, KW88b, KW91, KW94]. Our small coupling techniques were inspired by [GK83, GK86]. But we avoid the separate treatment of small and large field configurations by the use of a suitably weighted norm [P90, P93, Alb91].

This paper is organized as follows. In Section 2, we recall the hierarchical renormalization group and the role of the Gaussian fixed point in the construction of an invariant Banach space. In Section 3, we recall the trivial fixed point, its spectrum, and the cumulant expansion. In Section 4, we recall the formal perturbation expansion for the solution of (I), (II), and (III). In Section 5, we introduce a sequence of new interpolation formulas. They provide the tools for our estimates of non-perturbative errors. In Section 6, we estimate the error of the linear approximation by means of a complex Cauchy bound. In Section 7, we compute a quadratic fixed point of $R \times \delta^\star$. This quadratic fixed point serves to define the Banach space in which we look for a non-quadratic fixed point. In the Sections 8 and 9, we recall the contraction mapping principle. In Section 10, we present a real stability bound for the linear approximation. This bound is then generalized beyond the linear approximation in Section 11. We prove a stability bound for the perturbative approximants of order three, five, and seven by showing the dominance of tree contributions. In the Sections 12 and 13, we estimate the error of the perturbative fixed point of $R \times \delta^\star$. In conjunction with the results of the Sections 8 and 9, this completes the construction. For the seventh order approximant, we prove the contraction property for dimensions $D < 28/9$, with certain restrictions. Our perturbative approximant $Z_1(\phi, g)$ becomes singular in $D = 3$ dimensions. This case is therefore excluded in the sequel. However, this is not a problem of the contraction mapping method but rather a problem of finding a sufficiently accurate approximate solution. Such an approximate solution can also be found in $D = 3$ dimensions, namely by double perturbation theory in g and $g^2 \log(g)$. Since this case requires a special stability analysis with slight additional complications, we postpone it to a separate article. This article ends with a few conclusions and an outlook in Section 14. A few useful formulas are collected in the appendix.

2. HIERARCHICAL RENORMALIZATION GROUP

Consider the hierarchical renormalization group as a model for asymptotic freedom beyond perturbation theory [GK82, GK83, GK85a, GK86, P90, Alb91].

2.1. Hierarchical Renormalization Group Transformation

The hierarchical renormalization group is a semigroup of transformations generated by the following non-linear operator R. Let R be given by

$$R(Z)(\psi) = \int d\mu_\gamma(\zeta)\, Z(\beta\psi + \zeta)^\alpha \tag{4}$$

where α, β, and γ are real parameters, and where

$$d\mu_\gamma(\zeta) = (2\pi\gamma)^{-1/2}\, e^{-\zeta^2/2\gamma}\, d\zeta \tag{5}$$

is the Gaussian measure on $\mathbb{R}$ with mean zero and covariance γ. Let

$$\alpha = L^D, \qquad \beta = L^{1-D/2}, \qquad \gamma = 1 - L^{2-D} \tag{6}$$

with real parameters L and D; with $L > 1$ and certain restrictions on D. The meaning of L is that of a scale, the meaning of D is that of a dimension. Let α be integer valued, say $L = 2$ unless differently stated. Thereby, we circumvent multi-valued functions without loss of physical generality.

The literature on the hierarchical renormalization group usually deals with the normalized transformation

$$R_0(Z)(\psi) = \frac{R(Z)(\psi)}{R(Z)(0)} \tag{7}$$

The technical reason is that the normalization constant tends to grow exponentially upon iteration of (4). Our method will avoid this kind of instability without the division of (7).

The definition (4) of R calls to be supplemented by a suitable space of functions Z. A prototype of which is the following Banach space.

Let b be a real constant with $b > 0$. Let $\mathbb{B}$ be the complex strip

$$\mathbb{B} = \{\phi \in \mathbb{C} \mid |\Im(\phi)| \leqslant b\} \tag{8}$$

Consider then the Banach space of complex valued analytic functions Z: $\mathbb{B} \to \mathbb{C}$ with the following properties:

(I) For all $\phi \in \mathbb{B}$, let $Z(-\phi) = Z(\phi)$. We thus restrict our attention to $\mathbb{Z}_2$-symmetric functions.

(II) Let c be another real constant, $c > -1/\gamma$, and let $\|Z\|_{\infty, c}$ be the norm given by

$$\|Z\|_{\infty, c} = \sup_{\phi \in \mathbb{B}} |Z(\phi)\, e^{(c/2)\phi^2}| \tag{9}$$

Let Z be bounded in this norm, $\|Z\|_{\infty, c} < \infty$; consequently, the Gaussian integral in (4) becomes well defined.

Our choice of a function space will be adjusted according to our needs as we proceed, with this prototype in our mind. Another prototype is the analogous Banach space (with $b = 0$) of real valued continuous functions Z: $\mathbb{R} \to \mathbb{R}$. Both kinds of Banach spaces turn the analysis of (7) into a well defined mathematical problem. To represent an interaction Boltzmann weight of a Statistical Mechanical system, we should also impose the following positivity condition.

(III) For all $\phi \in \mathbb{R}$, let $Z(\phi) \in \mathbb{R}$ and $Z(\phi) > 0$; and thus $Z(\phi) = e^{-V(\phi)}$ with $V(\phi) \in \mathbb{R}$ and $V(-\phi) = V(\phi)$.

However, positive functions of this kind form a subset but not a linear subspace of our Banach space. To have Banach space theory available, we will analyze our fixed point problem in the general linear setting (including non-positive functions). The positivity of the outcome will be analyzed a posteriori.

2.2. Gaussian Fixed Point

Basic insight about R is gained from the transformation of Gauss functions. Gauss functions are transformed to Gauss functions according to

$$R\colon Z(\phi) = A e^{-b\phi^2/2} \mapsto R(Z)(\psi) = A' e^{-b'\psi^2/2} \tag{10}$$

with transformed parameters

$$A' = (1 + \alpha\gamma b)^{-1/2} A^{\alpha}, \qquad b' = \frac{\alpha\beta^2 b}{1 + \alpha\gamma b} \tag{11}$$

Besides a trivial fixed point $A_{UV}=1$, $b_{UV}=0$, one finds a quadratic fixed point

$$A_{HT}=(\alpha\beta^2)^{1/2(\alpha-1)}, \qquad b_{HT}=\frac{\alpha\beta^2-1}{\alpha\gamma} \tag{12}$$

The trivial fixed point serves as ultraviolet fixed point in ϕ^4-theory in dimensions $D<4$. The quadratic fixed point serves as a high temperature fixed point in any dimension. Notice that $\alpha\beta^2=L^2$ is independent of D.

2.3. Iterating Bound

From the quadratic fixed point, we obtain an iterating bound. It suggests a distinguished Banach space, where the unit ball centered at the origin is invariant. The significance of this Banach space is that it suitable for an infinite iteration of renormalization group transformations.

It is constructed from the following complex bound. Suppose that

$$|Z(\phi)| \leqslant A(\Re(\phi))\, B(\Im(\phi)) \tag{13}$$

for some positive bounded functions A and B, where $\Re(\phi)$ and $\Im(\phi)$ denotes the real and imaginary part of $\phi\in\mathbb{C}$ respectively. Then, if it exists,

$$|R(Z)(\psi)| \leqslant A'(\Re(\psi))\, B'(\Im(\psi)) \tag{14}$$

with

$$A'(\Re(\psi))=R(A)(\Re(\psi)), \qquad B'(\Im(\psi))=B(\beta\Im(\psi))^{\alpha} \tag{15}$$

Suppose that A and B are made such that

$$A'(\Re(\psi))\leqslant A(\Re(\psi)), \qquad B'(\Im(\psi))\leqslant B(\Im(\psi)) \tag{16}$$

Then it follows that $R(Z)$ satisfies the bound (13) if Z satisfies the bound (13). The quadratic fixed point provides a suitable function A.

$$A(\Re(\phi))=Z_{HT}(\Re(\phi)), \qquad B(\Im(\phi))=e^{c\,|\Im(\phi)|^q} \tag{17}$$

with parameters c and q such that $c>0$ and

$$\alpha\beta^q=L^{D+q(1-D/2)}=1, \qquad q=\frac{2D}{D-2} \tag{18}$$

yield a fixed point of (15). This pair of functions A and B satisfies in particular the bound (16).

The operator R preserves the bound (13). Any space consisting of functions, which obey (13) and whose other properties are also preserved by R, is mapped to itself. In particular, we can choose the Banach space of analytic functions on the complex strip $\mathbb{B}$ with the norm

$$\|Z\|_{\infty, A, B} = \sup_{\phi \in \mathbb{B}} \left| \frac{Z(\phi)}{A(\Re(\phi))\, B(\Im(\phi))} \right| \tag{19}$$

In this Banach space we can consider subsets or subspaces of functions with additional properties. The unit ball in this Banach space, consisting of functions such that

$$\|Z\|_{\infty, A, B} \leqslant 1 \tag{20}$$

is an invariant subset. We remark that the operator R preserves analyticity. If Z is an analytic function on $\mathbb{B}$ with finite norm (19), then also $R(Z)$ is an analytic function on $\mathbb{B}$. For this reason, we can restrict our attention to analytic functions.

The origin of these statements is the renormalization of ϕ^4-theory. There one studies the renormalization group flow with initial value

$$Z_{\nu,\mu,\lambda}(\phi) = \mathrm{e}^{-V_{\nu,\mu,\lambda}(\phi)}, \qquad V_{\nu,\mu,\lambda}(\phi) = \nu + \frac{\mu}{2} :\phi^2:_v + \frac{\lambda}{4!} :\phi^4:_v \tag{21}$$

A natural question to pose is which properties of (21) are preserved under the action of R. Analyticity in ϕ is such a property. Other properties require a refined analysis which is the subject of this paper.

3. TRIVIAL FIXED POINT

The trivial fixed point $Z_{UV}(\phi) = 1$ is not an element of this Banach space, and in particular not an element of the unit ball. We will study perturbations in a cone emerging from the trivial fixed point. Their analysis requires a modification of the norm (19), which encorporates a flowing parameter, a running coupling. This section serves to setup basic notations on an informal level.

3.1. Linearized Renormalization Group

The derivative of the operator R at a function Z defines a linear operator $D_Z R$ given by

$$D_Z R(\mathcal{O})(\psi) = \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right]_{\varepsilon=0} R(Z+\varepsilon\mathcal{O})(\psi)$$

$$= \alpha \int \mathrm{d}\mu_\gamma(\zeta)\, Z(\beta\psi+\zeta)^{\alpha-1}\, \mathcal{O}(\beta\psi+\zeta) \tag{22}$$

At the trivial fixed point $Z_{UV}(\phi)=1$ it becomes a rescaled Gaussian convolution

$$D_1 R(\mathcal{O})(\psi) = \alpha \int \mathrm{d}\mu_\gamma(\zeta)\, \mathcal{O}(\beta\psi+\zeta) = \alpha\langle \mathcal{O}\rangle_{\gamma,\beta\psi} \tag{23}$$

This linear operator $D_1 R$ is diagonalized by normal ordering. The normal ordered monomials $:\phi^n:_v = P_n(\phi, v)$, with normal ordering covariance v, are defined to be generated by the Gauss function

$$P(z, \phi, v) = \mathrm{e}^{z\phi - vz^2/2} = \sum_{n=0}^{\infty} \frac{z^n}{n!} P_n(\phi, v) \tag{24}$$

They are thus rescaled Hermite polynomials

$$P_n(\phi, v) = \left(\frac{v}{2}\right)^{n/2} \mathrm{H}_n\left(\frac{\phi}{\sqrt{2v}}\right) \tag{25}$$

The generating function (24) transforms under $D_1 R$ according to

$$D_1 R(P)(z, \psi, v) = \alpha P(z, \beta\psi, v-\gamma) = \alpha P(\beta z, \psi, \beta^{-2}(v-\gamma)) \tag{26}$$

It follows that

$$D_1 R(P_n)(\psi, v) = \alpha\beta^n P_n(\psi, \beta^{-2}(v-\gamma)) \tag{27}$$

When the normal ordering covariance v equals its fixed point value

$$v = \frac{\gamma}{1-\beta^2} = \frac{\gamma}{1-L^{2-D}} \overset{(6)}{=} 1 \tag{28}$$

the normal ordered monomials $P_n(\phi, v)$ turn into eigenfunctions of $D_1 R$. Their eigenvalues are

$$\alpha\beta^n = L^{\sigma_n}, \qquad \sigma_n = D + n\left(1-\frac{D}{2}\right) \tag{29}$$

Table I. Normal Ordered Monomials

$P_0(\phi, v) = 1$	$\sigma_0 = D$
$P_2(\phi, v) = \phi^2 - v$	$\sigma_2 = 2$
$P_4(\phi, v) = \phi^4 - 6v\phi^2 + 3v^2$	$\sigma_4 = 4 - D$
$P_6(\phi, v) = \phi^6 - 15v\phi^4 + 45v^2\phi^2 - 15v^3$	$\sigma_6 = 6 - 2D$
$P_8(\phi, v) = \phi^8 - 28v\phi^6 + 210v^2\phi^4 - 420v^3\phi^2 + 105v^4$	$\sigma_6 = 8 - 3D$
$P_{10}(\phi, v) = \phi^{10} - 45v\phi^8 + 630v^2\phi^6 - 3150v^3\phi^4 + 4725v^4\phi^2 - 945v^5$	$\sigma_{10} = 10 - 4D$

The system of eigenfunctions $\{P_n(\phi, v)\}_{n \in \mathbb{N}}$ forms a basis of $L_2(\mathbb{R}, d\mu_v(\phi))$. We will restrict our attention to the non-linear corrections in the case of one particular eigenfunction. The full spectral theory of $D_1 R$ will not be needed for this. The explicit form of the even eigenfunctions is

$$P_{2n}(\phi, v) = \sum_{m=0}^{n} P_{2n,2m}(v)\, \phi^{2(n-m)}, \qquad P_{2n,2m}(v) = \left(-\frac{v}{2}\right)^m \frac{(2n)!}{m!\,(2(n-m))!} \tag{30}$$

The first few of them are collected in Table I.

3.2. Cumulant Expansion

Let $Z(\phi) = e^{-V(\phi)}$ and $R(Z)(\psi) = e^{-T(V)(\psi)}$. The cumulant expansion is an expansion of $T(V)$ in powers of V. It reads

$$\begin{aligned} T(V)(\psi) &= -\log\left(\int d\mu_\gamma(\zeta)\, e^{-\alpha V(\beta\psi + \zeta)}\right) \\ &= \sum_{m=1}^{n} \frac{(-1)^{m+1}}{m!} \langle [\alpha V;]^m \rangle^T_{\gamma,\beta\psi,0} \\ &\quad - \frac{(-1)^{n+1}}{n!} \int_0^1 dt (1-t)^n \langle [\alpha V;]^{n+1} \rangle^T_{\gamma,\beta\psi,t} \end{aligned} \tag{31}$$

where $\langle \cdot \rangle^T$ denote the cumulants to the moments $\langle \cdot \rangle$, given by

$$\langle \mathcal{O} \rangle_{\gamma,\beta\psi,t} = \frac{\int d\mu_\gamma(\zeta)\, e^{-t\alpha V(\beta\psi + \zeta)} \mathcal{O}(\beta\psi + \zeta)}{\int d\mu_\gamma(\zeta)\, e^{-t\alpha V(\beta\psi + \zeta)}} \tag{32}$$

The cumulants of lowest orders are collected in Table II.

Table II. Cumulants

$\langle V\rangle_{\alpha'}^{T}=\langle V\rangle_{\alpha'}$
$\langle V;V\rangle_{\alpha'}^{T}=\langle V^2\rangle_{\alpha'}-\langle V\rangle_{\alpha'}^{2}$
$\langle V;V;V\rangle_{\alpha'}^{T}=\langle V^3\rangle_{\alpha'}-3\langle V^2\rangle_{\alpha'}\langle V\rangle_{\alpha'}^{2}+2\langle V\rangle_{\alpha'}^{3}$
$\langle V;V;V;V\rangle_{\alpha'}^{T}=\langle V^4\rangle_{\alpha'}-4\langle V^3\rangle_{\alpha'}\langle V\rangle_{\alpha'}-3\langle V^2\rangle_{\alpha'}^{2}+12\langle V^2\rangle_{\alpha'}\langle V\rangle_{\alpha'}^{2}-6\langle V\rangle_{\alpha'}^{4}$

The leading term in the cumulant expansion for T is the linearized renormalization group $D_1 R$. For $n=1$, Eq. (31) becomes

$$T(V)(\psi)=D_1 R(V)(\psi)-\int_0^1 \mathrm{d}t\,(1-t)\,\alpha^2\langle(V-\langle V\rangle_{\gamma,\beta\psi,t})^2\rangle_{\gamma,\beta\psi,t} \tag{33}$$

Observe that the non-linear contribution is always negative. The cumulant expansion is suitable for the computation of T provided that V is bounded. It is also the basis of formal perturbation theory.

We will consider perturbations where $V(\phi)$ is a polynomial in ϕ, and therefore unbounded. In this situation, the cumulant expansion applies only in a small field region. See [GK82, GK83, GK86]. To avoid the separate treatment of small and large field configurations, we will use an exponentiated version of the cumulant expansion in conjunction with estimates, which are true both for small and for large fields.

4. PERTURBATIVE ϕ^4-TRAJECTORY

In dimensions $D<4$, the eigenfunction $P_4(\phi,v)$ of $D_1 R$ defines an unstable perturbation of the trivial fixed point. Its eigenvalue is $\alpha\beta^4=L^{\sigma_4}$ with $\sigma_4=4-D$. Consider the renormalization problem to construct a renormalization invariant curve which emerges from the trivial fixed point tangent to $P_4(\phi,v)$, and which is parametrized by a running coupling g with linear step β-function δ. In this section, we recall its solution in formal perturbation theory in g from [RW96].

4.1. Fixed-Point Problem

Consider the renormalization problem to construct a curve $V(\phi,g)$ of potentials, parametrized by g, with the following properties:

$$V(\phi,g)=\frac{g}{4!}P_4(\phi,v)+O(g^2) \tag{34}$$

and

$$T(V)(\psi, \delta g) = V(\psi, g), \qquad \delta = \frac{1}{\alpha\beta^4} \tag{35}$$

In other words, we look for a fixed point V of the transformation $T \times \delta^\star$, the renormalization group composed with an inverse renormalization flow of g with linear step β-function.

4.2. Perturbation Theory

Equations (34) and (35) have a unique solution in the space of formal power series in g with polynomial coefficients, except at certain special dimensions [RW96]. We write it in the form

$$V(\phi, g) = \sum_{r=1}^{\infty} V^{(r)}(\phi) \frac{g^r}{r!}, \qquad V^{(r)}(\phi) = \sum_{n=0}^{r+1} \frac{V_{2n}^{(r)}}{(2n)!} P_{2n}(\phi, v) \tag{36}$$

A ϕ-independent normalization constant is included. Equation (35) yields a recursion relation for the coefficients $V_{2n}^{(r)}$, which determines them inductively on the order r. This recursion relation, has the following explicite form. From the cumulant expansion we deduce that, in the sense of formal power series in g,

$$\begin{aligned} T(V)(\psi, g) &= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!} \langle [\alpha V(\cdot, g);]^i \rangle_{\gamma, \beta\psi}^{T} \\ &= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!} \sum_{s_1=1}^{\infty} \frac{g^{s_1}}{s_1!} \cdots \sum_{s_i=1}^{\infty} \frac{g^{s_i}}{s_i!} \langle \alpha V^{(s_1)};...; \alpha V^{(s_i)} \rangle_{\gamma, \beta\psi}^{T} \\ &= \sum_{r=1}^{\infty} \frac{g^r}{r!} \sum_{i=1}^{r} \frac{(-1)^{i+1}}{i!} \sum_{s_1=1}^{r-i+1} \sum_{s_2=1}^{r-s_1-i+2} \cdots \sum_{s_{i-1}=1}^{r-\sum_{j=1}^{i-2}-1} \frac{r!}{\prod_{j=1}^{i} s_j!} \\ &\quad \times \langle \alpha V^{(s_1)};...; \alpha V^{(s_i)} \rangle_{\gamma, \beta\psi}^{T} \end{aligned} \tag{37}$$

where $s_i = r - \sum_{j=1}^{i-1} s_j$. We separate out the term which depend linearly on V on the right hand side of Eq. (37) to obtain a representation

$$T(V)(\phi, g) = \sum_{r=1}^{\infty} \frac{g^r}{r!} \{\alpha\langle V^{(r)} \rangle_{\gamma, \beta\psi} + K(V)^{(r)}(\psi)\} \tag{38}$$

where $K(V)^{(r)}$ depends on $V^{(s)}$ with $s<r$ only. To every order r of perturbation theory, the order r contribution on the right hand side of (37) is transformed according to the linearized renormalization group. Equation (35) equippes us with a linear equation

$$V^{(r)}(\psi)-\alpha\delta^{r}\langle V^{(r)}\rangle_{\gamma,\beta\psi}=\delta^{r}K(V)^{(r)}(\psi) \tag{39}$$

whose left hand side is brought to diagonal form by normal ordering. The right hand side of (39) comes out as a polynomial

$$K(V)^{(r)}(\psi)=\sum_{n=0}^{r+1}\frac{K(V)_{2n}^{(r)}}{(2n)!}P_{2n}(\psi,v) \tag{40}$$

with coefficients

$$K(V)_{2n}^{(r)}=v^{-2n}\int d\mu_{v}(\psi)\,P_{2n}(\psi,v)\,K(V)^{(r)}(\psi) \tag{41}$$

Therefore, Eq. (39) furnishes a system of linear equations

$$(1-\alpha\beta^{2n}\delta^{r})\,V_{2n}^{(r)}=\delta^{r}K(V)_{2n}^{(r)} \tag{42}$$

for the coefficients $V_{2n}^{(r)}$. Equation (42) determines $V_{2n}^{(r)}$ unless

$$\alpha\beta^{2n}\delta^{r}=L^{D+n(2-D)-r(4-D)}=1 \tag{43}$$

This situation was called an (r,n)-resonance in [RW96]. Resonances occur at a discrete series of special dimensions. For instance, in three dimensions both a $(2,1)$-resonance and a $(3,0)$-resonance occurs. Resonances affect only the perturbative part of our construction. They can be resolved by $\log(g)$ corrections. For simplicity, we will restrict our attention to the case of non-resonant dimensions in this paper.

To summarize, (34) and (35) have a unique solution in the sense of formal power series in g. This solution is the perturbative ϕ^4-trajectory. The aim of this paper is to construct a nonperturbative version of it. The recursion relation given by (42) can be iterated to high orders by means of computer algebra.

Table III. Perturbative Renormalization Group

$T^{(1)}(V)(\psi)=\alpha\langle V^{(1)}\rangle_{\gamma,\beta\psi}$
$T^{(2)}(V)(\psi)=\alpha\langle V^{(2)}\rangle_{\gamma,\beta\psi}-\alpha^{2}\langle V^{(1)};V^{(1)}\rangle_{\gamma,\beta\psi}^{T}$
$T^{(3)}(V)(\psi)=\alpha\langle V^{(3)}\rangle_{\gamma,\beta\psi}-3\alpha^{2}\langle V^{(1)};V^{(2)}\rangle_{\gamma,\beta\psi}^{T}+\alpha^{3}\langle V^{(1)};V^{(1)};V^{(1)}\rangle_{\gamma,\beta\psi}^{T}$

5. INTERPOLATION FORMULAS

Interpolation formulas are basic tools for estimates of nonperturbative contributions generated by Gaussian convolutions. In this section, we recall a basic interpolation formula, and introduce thereafter a sequence of improved interpolation formulas adapted to our fixed point problem. Eq. (4) consists of a Gaussian convolution composed with a scale transformation parametrized by α and β. In this section, we investigate interpolations for the Gaussian convolution without scale transformation (with $\alpha=\beta=1$).

5.1. Covariance Interpolation

The most prominent interpolation formula follows from Gaussian integration by parts. See [GJ87]. Differentiate

$$Z(\psi, s)=\int \mathrm{d}\mu_{s\gamma}(\zeta)\, Z(\psi+\zeta) \tag{44}$$

with respect to the interpolation parameter s to obtain

$$\frac{\partial}{\partial s} Z(\psi, s)=\int \mathrm{d}\mu_{s\gamma}(\zeta)\, \frac{\gamma}{2} \frac{\partial^2}{\partial \zeta^2} Z(\psi+\zeta)=\frac{\gamma}{2} \frac{\partial^2}{\partial \psi^2} Z(\psi, s) \tag{45}$$

The interpolated Gaussian Convolution in (44) is thus the solution of a heat equation with initial condition $Z(\psi, 0)=Z(\psi)$. The interpolation parameter goes from zero to one.

Its basic use in mathematical renormalization theory is to establish Cauchy bounds of the following type. Suppose that $Z(\psi, s)$ is an analytic function of ψ, say in a strip around the real axis. Then it follows that

$$|Z(\psi, 1)-Z(\psi, 0)| \leqslant \frac{\gamma}{2} \int_0^1 \mathrm{d}s\left|\frac{\partial^2}{\partial \psi^2} Z(\psi, s)\right| \leqslant \frac{\gamma}{R^2} \sup_{|\chi|=R, s} |Z(\psi+\chi, s)| \tag{46}$$

A typical value of the Cauchy radius is $R=g^{-q}$, where g is a coupling constant, with $q=1/4$ in ϕ^4-theory [GK83, GK86, P90]. Provided that $Z(\psi, s)$ is bounded in a strip around the real axis, one gets an estimate of the order g^{2q} for the difference of boundary values.

5.2. Improved Interpolation

In the interpolation defined by (44), the Gaussian convolution is compared to the identity operation. Consider the following different interpolation, which combines the previous one with the cumulant expansion. Let $Z(\phi) = \mathrm{e}^{-V(\phi)}$, and

$$Z(\psi, s) = \int \mathrm{d}\mu_{s\gamma}(\zeta) \exp\left(- \int \mathrm{d}\mu_{(1-s)\gamma}(\zeta)\ V(\psi + \zeta + \xi) \right) \tag{47}$$

The interpolation parameter s runs from zero to one. The boundary values of the interpolation (47) are

$$Z(\psi, 0) = \exp\left(- \int \mathrm{d}\mu_{\gamma}(\xi)\ V(\psi + \xi) \right) \tag{48}$$

the exponentiated Gaussian convolution, and

$$Z(\psi, 1) = \int \mathrm{d}\mu_{\gamma}(\zeta) \exp(-V(\psi + \zeta)) \tag{49}$$

the ordinary Gaussian convolution. Differentiate (47) with respect to the interpolation parameter s to obtain

$$\begin{aligned}
\frac{\partial}{\partial s} Z(\psi, s) &= \int \mathrm{d}\mu_{s\gamma}(\zeta) \frac{\gamma}{2} \frac{\partial^2}{\partial \zeta^2} \exp\left(- \int \mathrm{d}\mu_{(1-s)\gamma}(\xi)\ V(\psi + \zeta + \xi) \right) \\
&\quad + \int \mathrm{d}\mu_{s\gamma}(\zeta) \exp\left(- \int \mathrm{d}\mu_{(1-s)\gamma}(\xi)\ V(\psi + \zeta + \xi) \right) \\
&\quad \times \int \mathrm{d}\mu_{(1-s)\gamma}(\xi) \frac{\gamma}{2} \frac{\partial^2}{\partial \xi^2}\ V(\psi + \zeta + \xi) \\
&= \frac{\gamma}{2} \int \mathrm{d}\mu_{s\gamma}(\zeta) \exp\left(- \int \mathrm{d}\mu_{(1-s)\gamma}(\xi)\ V(\psi + \zeta + \xi) \right) \\
&\quad \times \left\{ \frac{\partial}{\partial \psi} \int \mathrm{d}\mu_{(1-s)\gamma}(\xi)\ V(\psi + \zeta + \xi) \right\}^2
\end{aligned} \tag{50}$$

The term in curly brackets $\{\cdot\}$ in (50) is called a downstairs factor in mathematical renormalization theory. For polynomial potentials, downstairs factors are again polynomials and hence unbounded. Their large field growth is however compensated by the exponential decrease of the weight factor. This mechanism is called domination. To illustrate it, we remark that

$$\frac{\partial}{\partial s} Z(\psi, s) = 2\gamma \int \mathrm{d}\mu_{s\gamma}(\zeta) \left\{ \frac{\partial}{\partial \psi} \exp\left(-\frac{1}{2} \int \mathrm{d}\mu_{(1-s)\gamma}(\xi)\ V(\psi + \zeta + \xi) \right) \right\}^2 \tag{51}$$

is ready for a Cauchy estimate on the exponential.

Another interpolation formula, where not only the fluctuation covariance but also certain running couplings are interpolated, appeared in [P90].

5.3. Higher Interpolation Formulas

The interpolation (47) is the first in a series of formulas, where the Gaussian convolution is compared to the exponentiated cumulant expansion of any finite order. Consider the interpolated Gaussian moments

$$\langle \mathcal{O} \rangle_{(1-s)\gamma, \psi} = \int d\mu_{(1-s)\gamma}(\zeta)\, \mathcal{O}(\psi + \zeta) \tag{52}$$

and their associated cumulants

$$\langle [\mathcal{O};]^n \rangle^T_{(1-s)\gamma, \psi} = \left[\frac{\partial^n}{\partial t^n}\right]_{t=0} \log\langle \exp(t\mathcal{O})\rangle_{(1-s)\gamma, \psi} \tag{53}$$

For $n \geqslant 1$, define

$$T_s(V)(\psi) = \sum_{m=1}^{n} \frac{(-1)^{m+1}}{m!} \langle [V;]^m \rangle^T_{(1-s)\gamma, \psi} \tag{54}$$

Consider then the interpolation given by

$$R_s(Z)(\psi) = \langle \exp(-T_s(V))\rangle_{s\gamma, \psi} \tag{55}$$

between

$$R_0(Z)(\psi) = e^{-T_0(V)(\psi)} \tag{56}$$

and

$$R_1(Z)(\psi) = R(Z)(\psi) \tag{57}$$

where R denotes the Gaussian convolution without scale factors α and β. The interpolation of (47) is the first of this series, with $n = 1$. Differentiate (55) with respect to the interpolation parameter s to obtain

$$\frac{\partial}{\partial s} R_s(Z)(\psi) = \frac{\gamma}{2}\left\langle e^{-T_s(V)} \frac{1}{n!} \int_0^1 dt(1-t)^n \frac{\partial^{n+1}}{\partial t^{n+1}} \left\{\frac{\partial}{\partial \psi} T_s(tV)\right\}^2 \right\rangle_{s\gamma, \psi} \tag{58}$$

The parameter integral under the expectation value serves as a projector

$$\frac{1}{n!}\int_0^1 \mathrm{d}t(1-t)^n \frac{\partial^{n+1}}{\partial t^{n+1}} t^m = \begin{cases} 0, & m \leqslant n \\ 1, & m > n \end{cases} \tag{59}$$

The downstairs factor in (58) is therefore of the order V^{n+1}, with two field derivatives and truncated expectations. Notice that the truncated expectations select connected contributions. One can think of the right hand side of (58) as a non-perturbative contraction. Equation (58) follows from

$$\begin{aligned} \frac{\partial}{\partial s} R_s(Z)(\psi) &= \left\langle \frac{\gamma}{2}\frac{\partial^2}{\partial\psi^2} \mathrm{e}^{-T_s(V)} \right\rangle_{s\gamma,\psi} - \left\langle \mathrm{e}^{-T_s(V)} \frac{\partial}{\partial s} T_s(V) \right\rangle_{s\gamma,\psi} \\ &= \left\langle \mathrm{e}^{-T_s(V)} \left\{ -\left(\frac{\partial}{\partial s} + \frac{\gamma}{2}\frac{\partial^2}{\partial\psi^2}\right) T_s(V) + \frac{\gamma}{2}\left(\frac{\partial}{\partial\psi} T_s(V)\right)^2 \right\} \right\rangle_{s\gamma,\psi} \end{aligned} \tag{60}$$

and

$$\begin{aligned} -\left(\frac{\partial}{\partial s} + \frac{\gamma}{2}\frac{\partial^2}{\partial\psi^2}\right) T_s(V)(\psi) + \frac{\gamma}{2}\left(\frac{\partial}{\partial\psi} T_s(V)(\psi)\right)^2 \\ = \frac{1}{n!}\int_0^1 \mathrm{d}t(1-t)^n \frac{\partial^{n+1}}{\partial t^{n+1}} \frac{\gamma}{2}\left(\frac{\partial}{\partial\psi} T_s(tV)(\psi)\right)^2 \end{aligned} \tag{61}$$

Consider the case $n=2$ as an illustration of the cancellation which is happening here. There we have that

$$\begin{aligned} T_s(V)(\psi) &= \langle V \rangle_{(1-s)\gamma,\psi} - \tfrac{1}{2}\langle V; V \rangle^T_{(1-s)\gamma,\psi} \\ &= \langle V \rangle_{(1-s)\gamma,\psi} - \tfrac{1}{2}\langle V^2 \rangle_{(1-s)\gamma,\psi} + \tfrac{1}{2}\langle V \rangle^2_{(1-s)\gamma,\psi} \end{aligned} \tag{62}$$

The s derivative of (62) is given by

$$\begin{aligned} \frac{\partial}{\partial s} T_s(V)(\psi) &= -\frac{\gamma}{2}\frac{\partial^2}{\partial\psi^2}\left(\langle V \rangle_{(1-s)\gamma,\psi} - \frac{1}{2}\langle V^2 \rangle_{(1-s)\gamma,\psi}\right) \\ &\quad - \langle V \rangle_{(1-s)\gamma,\psi} \frac{\gamma}{2}\frac{\partial^2}{\partial\psi^2} \langle V \rangle_{(1-s)\gamma,\psi} \\ &= -\frac{\gamma}{2}\frac{\partial^2}{\partial\psi^2} T_s(V)(\psi) + \frac{\gamma}{2}\left(\frac{\partial}{\partial\psi}\langle V \rangle_{(1-s)\gamma,\psi}\right)^2 \end{aligned} \tag{63}$$

The second term on the right hand side of (63) cancels the order V^2 term on the right hand side of

$$-\left(\frac{\partial}{\partial s}+\frac{\gamma}{2}\frac{\partial^2}{\partial\psi^2}\right)T_s(V)(\psi)+\frac{\gamma}{2}\left(\frac{\partial}{\partial\psi}T_s(V)(\psi)\right)^2$$
$$=\frac{\gamma}{2}\left(\frac{\partial}{\partial\psi}T_s(V)(\psi)\right)^2-\frac{\gamma}{2}\left(\frac{\partial}{\partial\psi}\langle V\rangle_{(1-s)\gamma,\psi}\right)^2 \tag{64}$$

which gives (58) in the case $n=2$. Throughout this section we have assumed that the Gaussian expectations exist and define differentiable functions of their parameters. This of course has to be verified in each particular theory.

6. LINEAR APPROXIMATION I

This section contains a nonperturbative bound on the error up to which the linear approximation, defined by

$$Z(\phi, g)=e^{-V(\phi, g)}, \qquad V(\phi, g)=\frac{g}{4!}P_4(\phi, v) \tag{65}$$

is a fixed point of $R\times\delta^\star$, where $\delta=1/\alpha\beta^4$. We shall use the first interpolation formula (47) (with an additional scale transformation) to derive a standard bound from constructive renormalization theory on the hierarchical renormalization group

6.1. Interpolation Formula

With the flow function $\delta=1/\alpha\beta^4$, the potential (65) is a fixed point of $D_1 R\times\delta^\star$; it satisfies

$$\alpha\langle V(\cdot, \delta g)\rangle_{\gamma, \beta\psi}=\alpha\delta\frac{g}{4!}\langle P_4(\cdot, v)\rangle_{\gamma, \beta\psi}=\alpha\beta^4\delta\frac{g}{4!}P_4(\psi, v)=V(\psi, g) \tag{66}$$

Consider therefore the interpolation formula

$$R_s(Z)(\psi, g)=\int d\mu_{s\gamma}(\zeta)\exp\left(-\alpha\int d\mu_{(1-s)\gamma}(\zeta)\,V(\beta\psi+\zeta+\xi, \delta g)\right) \tag{67}$$

from our interpolation toolbox. It interpolates between

$$R_0(Z)(\psi, g) = \exp\left(-\alpha \int d\mu_\gamma(\xi)\ V(\beta\psi + \xi, \delta g)\right)$$
$$= \exp(-(D_1 R \times \delta^\star)(V)(\psi, g)) \tag{68}$$

and

$$R_1(Z)(\psi, g) = \int d\mu_\gamma(\zeta) \exp(-\alpha V(\beta\psi + \zeta, \delta g)) = (R \times \delta^\star)(Z)(\psi, g) \tag{69}$$

For the linear approximation (65), we therefore have that

$$(R \times \delta^\star - \mathrm{i}d)(Z)(\psi, g) = \int_0^1 ds \frac{\partial}{\partial s} R_s(Z)(\psi, g) \tag{70}$$

This equation is suited to bound the error of the linear approximation. According to (50) and (51), the parameter derivative is here given by

$$\frac{\partial}{\partial s} R_s(Z)(\psi, g) = \frac{\gamma}{2} \int d\mu_{s\gamma}(\zeta)\, e^{-\alpha \int d\mu_{(1-s)\gamma}(\xi)\ V(\beta\psi + \zeta + \xi, \delta g)}$$
$$\times \left\{ \frac{\partial}{\partial(\beta\psi)} \alpha \int d\mu_{(1-s)\gamma}(\xi)\ V(\beta\psi + \zeta + \xi, \delta g) \right\}^2$$
$$= 2\gamma \int d\mu_{s\gamma}(\xi) \left\{ \frac{\partial}{\partial(\beta\psi)} e^{-\alpha/2 \int d\mu_{(1-s)\gamma}(\xi)\ V(\beta\psi + \zeta + \xi, \delta g)} \right\}^2 \tag{71}$$

By elementary analysis, (70) and (71) are well defined and valid identities for the case of (65).

6.2. Large-Field Bound

The interpolated Gaussian convolution in the exponent of the weight factor in Eq. (50) changes only the normal ordering covariance according to the formula (27). Abbreviate $\beta\psi + \zeta = \phi$, to obtain

$$\alpha \int d\mu_{(1-s)\gamma}(\xi)\ V(\phi + \xi, \delta g) = \alpha\delta \frac{g}{4!} P_4(\phi, v_s), \qquad v_s = v - (1-s)\gamma \tag{72}$$

We have the following stability bound on the real part of the normal ordered monomial $P_4(\phi, v_s)$ as an entire function of $\phi \in \mathbb{C}$:

There exist positive constants $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that for all $\phi \in \mathbb{R}$ and $\chi \in \mathbb{C}$ and $s \in [0, 1]$ the following lower bound holds:

$$\Re(P_4(\phi + \chi, v_s)) \geqslant \frac{\phi^2}{2} - a\,|\chi|^4 - b \tag{73}$$

Large field bounds of this type were pioneered by [GK80, GK82, GK83, GK86]. Exactly this bound appeared in [P90]. A proof of (73) is given in Appendix 15.2.

6.3. Cauchy Estimate

Let us combine the stability bound (73) with the Cauchy estimate of ineq. (46) to obtain an upper bound on the modulus of the parameter derivative (71). The weight factor

$$\exp\left(-\frac{\alpha}{2}\int d\mu_{(1-s)\gamma}(\xi)\; V(\phi + \xi, \delta g)\right) = \exp\left(-\frac{\alpha\,\delta g}{2\cdot 4!}\, P_4(\phi, v_s)\right) \tag{74}$$

in Eq. (71) is an entire function and obeys the stability bound

$$\begin{aligned} &\left|\exp\left(-\frac{\alpha}{2}\int d\mu_{(1-s)\gamma}(\xi)\; V(\phi + \chi + \xi, \delta g)\right)\right| \\ &\qquad \leqslant \exp\left\{-\frac{\alpha\,\delta g}{2\cdot 4!}\left(\frac{\phi^2}{2} - a\,|\chi|^4 - b\right)\right\} \end{aligned} \tag{75}$$

for all $\phi \in \mathbb{R}$, $\chi \in \mathbb{C}$, and $s \in [0, 1]$. Consequently,

$$\begin{aligned} &\left|\frac{\partial}{\partial\phi}\exp\left(-\frac{\alpha}{2}\int d\mu_{(1-s)\gamma}(\xi)\; V(\phi + \xi, \delta g)\right)\right| \\ &\qquad \leqslant \frac{1}{R}\sup_{|\chi| = R}\left|\exp\left(-\frac{\alpha\,\delta g}{2\cdot 4!}\, P_4(\phi + \chi, v_s)\right)\right| \\ &\qquad \leqslant \frac{1}{R}\exp\left\{-\frac{\alpha\,\delta g}{2\cdot 4!}\left(\frac{\phi^2}{2} - aR^4 - b\right)\right\} \\ &\qquad \underbrace{=}_{R = g^{-1/4}}\; g^{1/4}\exp\left\{-\frac{\alpha\delta}{2\cdot 4!}\left(\frac{g\phi^2}{2} - a - bg\right)\right\} \end{aligned} \tag{76}$$

Therefrom we obtain an upper bound on the modulus of (71). It reads

$$\begin{aligned}
&\left|\frac{\partial}{\partial s} R_s(Z)(\psi, g)\right| \\
&\quad \leqslant 2\gamma \int d\mu_{s\gamma}(\zeta) \left|\frac{\partial}{\partial \phi} \exp\left(-\frac{\alpha}{2} \int d\mu_{(1-s)\gamma}(\zeta)\, V(\phi+\xi, \delta g)\right)\right|^2_{\phi=\beta\psi+\zeta} \\
&\quad \leqslant 2\gamma \exp\left(\frac{\alpha\delta}{4!}(a+bg)\right) g^{1/2} \int d\mu_{s\gamma}(\zeta) \exp\left(-\frac{\alpha\,\delta g}{2\cdot 4!}(\beta\psi+\zeta)^2\right) \\
&\quad = 2\gamma \exp\left(\frac{\alpha\,\delta}{4!}(a+bg)\right)\left(1+s\gamma\frac{\alpha\,\delta g}{4!}\right)^{-1/2} \\
&\qquad \times g^{1/2} \exp\left\{-\frac{\alpha\beta^2\,\delta g}{2\cdot 4!}\left(1+s\gamma\frac{\alpha\,\delta g}{4!}\right)^{-1} \psi^2\right\}
\end{aligned} \tag{77}$$

Notice that the decay constant of the Gaussian is proportional to

$$\alpha\beta^2\delta = \frac{\alpha\beta^2}{\alpha\beta^4} = \beta^{-2} = L^{D-2} \tag{78}$$

For $D > 2$, it can be made large by enlarging L, which is common practice in mathematical renormalization theory. We will not need this in our analysis. We are now ready to estimate the quality of the linear approximation.

6.4. Error Bound

Let Z be given by (65). For all $\psi \in \mathbb{R}$ and $g \geqslant 0$, the modulus of (70) satisfies the bound

$$\begin{aligned}
&|(\mathbb{R} \times \delta^{\star} - id)(Z)(\psi, g)| \\
&\quad \leqslant \int_0^1 ds \left|\frac{\partial}{\partial s} R_s(Z)(\psi, g)\right| \\
&\quad \leqslant 2\gamma \exp\left(\frac{\alpha\delta}{4!}(a+bg)\right) g^{1/2} \exp\left\{-\frac{\alpha\beta^2\,\delta g}{2\cdot 4!}\left(1+\gamma\frac{\alpha\,\delta g}{4!}\right)^{-1} \psi^2\right\}
\end{aligned} \tag{79}$$

We emphasize that this bound holds for large and small values of ψ. Unnecessary to point out that the error is small when g is small. The power $g^{1/2}$ will not suffice for all purposes below. It will have to be improved upon. Before entering this task, we will continue and see where we get with the bound we have.

7. QUADRATIC FIXED POINT

In this section, we will compute a g-dependent quadratic fixed point of the composed transformation

$$(R \times \delta^{\star})(Z)(\psi, g) = \int \mathrm{d}\mu_{\gamma}(\zeta)\, Z(\beta\psi + \zeta, \delta g)^{\alpha} \tag{80}$$

It will supply us with a norm, with respect to which we will construct the nonperturbative ϕ^4-fixed point. As we will see, our quadratic fixed point interpolates between the trivial fixed point Z_{UV} at $g = 0$ and the quadratic fixed point Z_{HT} at $g = \infty$. Equation (80) represents the ϕ^2 or high temperature trajectory.

7.1. Transformation of Gauss Functions

Gauss functions are transformed to Gauss functions by (80) according to the transformation law

$$Z(\phi, g) = A(g)\, \mathrm{e}^{-(b(g)/2)\,\phi^2} \mapsto Z'(\psi, g) = A'(g)\, \mathrm{e}^{-(b'(g)/2)\,\psi^2} \tag{81}$$

with

$$A'(g) = A(\delta g)^{\alpha}\,(1 + \alpha\gamma b(\delta g))^{-1/2}, \qquad b'(g) = \frac{\alpha\beta^2 b(\delta g)}{1 + \alpha\gamma b(\delta g)} \tag{82}$$

the immediate generalization of the g-independent transformation law given by (10) and (11).

7.2. Fixed Point Value of b

As a fixed point equation for the function $c(g) = 1/b(g)$ we find a linear difference equation

$$\alpha\beta^2 c(g) = c(\delta g) + \alpha\gamma \tag{83}$$

It has a one parametric set of solutions given by

$$c(g) = c_{HT} + C g^{-\rho}, \qquad c_{HT} = \frac{\alpha\gamma}{\alpha\beta^2 - 1}, \qquad \rho = \frac{2}{4 - D} \tag{84}$$

where C is the free parameter. We restrict our attention to the case when $D < 4$, and thus $\rho > 0$. The fixed point value of b is thus

$$b(g) = \frac{g^{\rho}}{C + c_{HT}\, g^{\rho}} \tag{85}$$

Its limit values are $b(0)=0$ and $b(\infty)=b_{HT}$. In other words, we have an interpolation between the trivial fixed point at $g=0$ and the high temperature fixed point at $g=\infty$.

7.3. Fixed point value of *A*

At this fixed point b, we have that

$$1+\alpha\gamma b(g)=\frac{C+\alpha\beta^2 c_{HT} g^\rho}{C+c_{HT} g^\rho}=\left(\frac{d(\delta^{-1}g)}{d(g)}\right)^2, \qquad d(g)=\sqrt{C+c_{HT} g^\rho} \tag{86}$$

We choose $C>0$ to avoid troubles with the root. The fixed point equation for A then becomes

$$A(g)=A(\delta g)^\alpha \frac{d(\delta g)}{d(g)} \tag{87}$$

We look for fixed points where $A(g)>0$. Exponentiate $A(g)=e^{a(g)}$ to obtain another linear difference equation given by

$$a(g)=\alpha a(\delta g)+\log\left(\frac{d(\delta g)}{d(g)}\right) \tag{88}$$

The general solution to this equation is

$$a(g)=C' g^{\rho'}+\frac{\alpha-1}{\alpha}\sum_{n=1}^{\infty}\alpha^{-n}\log\left(\frac{d(\delta^{-n}g)}{d(g)}\right), \qquad \rho'=\frac{D}{4-D} \tag{89}$$

Here C' is another free parameter at our disposal. We put $C'=0$. In this case we have that $A(0)=1$ and $A(\infty)=A_{HT}$, and also the normalization interpolates between the free fixed point and the high temperature fixed point.

7.4. Small-*g* Behavior

For small values of g, the function $a(g)$ has the following behavior as a function of g^ρ:

$$\log\left(\frac{d(\delta^{-n}g)}{d(g)}\right)=\frac{c_{HT}}{2C}(\delta^{-n\rho}-1)\, g^\rho+O((g^\rho)^2) \tag{90}$$

and thus

$$a(g) = a^{(1)} g^{\rho} + O((g^{\rho})^2), \qquad a^{(1)} = \frac{c_{HT}}{2C} \frac{\delta^{-\rho} - 1}{\alpha - \delta^{-\rho}} = \frac{c_{HT}}{2C} \frac{\alpha\beta^2 - 1}{\alpha - \alpha\beta^2} \tag{91}$$

There remains the free parameter C. We will not attempt to optimize our bounds by tuning C. Instead we choose $C = c_{HT}$ in the sequel.

7.5. Parameter-Dependent Norm

We have computed a Gaussian fixed point of the composed transformation of $R \times \delta^\star$. For the parameter values $C = c_{HT}$ and $C' = 0$, it takes the form

$$Z_{QU}(\phi, g) = \exp\left(a_{QU}(g) - \frac{b_{QU}(g)}{2} \phi^2 \right) \tag{92}$$

with coefficient functions

$$a_{QU} = \frac{1 - \alpha^{-1}}{2} \sum_{n=1}^{\infty} \alpha^{-n} \log\left(\frac{1 + (\delta^{-n} g)^{\rho}}{1 + g^{\rho}} \right), \qquad b_{QU}(g) = b_{HT} \frac{g^{\rho}}{1 + g^{\rho}} \tag{93}$$

where $\rho = 2/4 - D$. The analogy to the g independent case suggests the following norm. Let $g_0 > 0$ be a fixed number, a maximal value of g in all below estimates, optimally infinity. Let $\mathscr{Z} \subset \mathscr{C}(\mathbb{R} \times (0, g_0), \mathbb{R})$ be the Banach space of real valued continuous functions Z: $(\phi, g) \mapsto Z(\phi, g)$ with the following properties:

(I) $Z(\phi, g)$ is $\mathbb{Z}_2$-symmetric in ϕ,

$$Z(-\phi, g) = Z(\phi, g) \tag{94}$$

(II) $Z(\phi, g)$ is finite in the norm

$$\|Z\|_{\infty} = \sup_{g \in (0, g_0)} \|Z(\cdot, g)\|_{\infty, g}, \qquad \|Z(\cdot, g)\|_{\infty, g} = \sup_{\phi \in \mathbb{R}} \left| \frac{Z(\phi, g)}{Z_{QU}(\phi, g)} \right| \tag{95}$$

We notice the option to restrict g to (ε, g_0), with $\varepsilon > 0$ arbitrary small, in order to exclude the point $g = 0$ which is special in the below constructions. Furthermore, we notice the option to work on the hyperplanes of constant g. Positive functions with $Z(\phi, g)$ form a subset but not a linear subspace. Although only positive functions lead yield probability measures, we will use the full Banach space to analyze the fixed point problem. The fixed point will be shown to be positive in the sequel.

As in the g-independent case, we have an invariant space. Functions $Z \in \mathscr{Z}$ obey

$$|Z(\phi, g)| \leqslant \|Z\|_\infty Z_{QU}(\phi, g) \tag{96}$$

and consequently

$$|(R \times \delta^\star)(Z)(\psi, g)| \leqslant \|Z\|_\infty^\alpha Z_{QU}(\psi, g) \tag{97}$$

The interior of the unit ball, given by functions $Z \in \mathscr{Z}$ such that

$$\|Z\|_\infty < 1 \tag{98}$$

is consequently contracted to zero. The unit ball itself is invariant under $R \times \delta^\star$. It follows that the norm of a non-trivial fixed point is larger or equal to one.

8. INVARIANT CONE

We will look for a fixed point Z of $R \times \delta^\star$ in the Banach space $\mathscr{Z}$. Our strategy will be to write Z as a sum of an approximate fixed point Z_1 and a correction Z_2, and to choose Z_1 sufficiently accurately that the transformation of Z_2 contracts. The constants below C_i, σ_i, and g_i will be understood to be locally defined in this and the next section.

8.1. Splitting of Z

Given an approximate fixed point Z_1, for instance the linear approximation

$$Z_1(\phi, g) = \exp\left(-\frac{g}{4!} P_4(\phi, v)\right) \tag{99}$$

we can split Z in a sum $Z_1 + Z_2$, and consider the transformation of the correction term Z_2 alone. The correction Z_2 transforms according to

$$\begin{aligned}(R \times \delta^\star)(Z_1 + Z_2) &= (R \times \delta^\star)(Z_1) + R_{Z_1}(Z_2)\\ &= Z_1 + (R \times \delta^\star - \mathrm{i}d)(Z_1) + R_{Z_1}(Z_2)\\ &= Z_1 + \triangle(Z_1) + R_{Z_1}(Z_2)\end{aligned} \tag{100}$$

We find two contributions. The first contribution is the error up to which Z_1 is a fixed point of $R \times \delta^\star$. The correction Z_2 cannot be smaller than this error, and it will therefore be important to choose Z_1 as close to the fixed point as possible. The second contribution can be estimated in terms of its derivative, or linearization. It is given by

$$R_{Z_1}(Z_2) = (R \times \delta^\star)(Z_1 + Z_2) - (R \times \delta^\star)(Z_1)$$
$$= \int_0^1 \mathrm{d}s \frac{\partial}{\partial s} (R \times \delta^\star)(Z_1 + s\, Z_2) \tag{101}$$

The existence of this interpolation will follow from elementary analysis for all choices of Z_1 and Z_2 considered in the sequel.

For the hierarchical renormalization group (4), it takes the form

$$\frac{\partial}{\partial s} (R \times \delta^\star)(Z_1 + s\, Z_2)(\psi, g)$$
$$= \int \mathrm{d}\mu_\gamma(\zeta)\, \alpha\, \{Z_1(\beta\psi + \zeta, \delta g) + s\, Z_2(\beta\psi + \zeta, \delta g)\}^{\alpha - 1} Z_2(\beta\psi + \zeta, \delta g) \tag{102}$$

8.2. Estimate on $R_{Z_1}(Z_2)$

Suppose that we have the following bounds on the approximate fixed point Z_1 and its error $\triangle(Z_1)$.

Let C_1, C_2, σ_1, and g_1 be positive constants. Let Z_1 and $\triangle(Z_1)$ satisfy the bounds

$$|Z_1(\phi, g)| \leqslant C_1\, Z_{QU}(\phi, g), \qquad |\triangle(Z_1)(\phi, g)| \leqslant C_2 g^{\sigma_1} Z_{QU}(\phi, g) \tag{103}$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_1]$.

In the case of the linear approximation (99), both bounds (103) are valid. We will come back to this issue when we will look for an optimal value of σ'. The size of the correction Z_2 is at least that of $\triangle(Z_1)$. For small g, $\triangle(Z_1)$ is of the order g^{σ_1}. Let σ_2 be another positive exponent, which is slightly smaller than σ_1, and assume that Z_2 is of the order g^{σ_2} for small g.

Let C_3, σ_2, and g_2 be positive constants, with $\sigma_2 < \sigma_1$. Let Z_2 satisfy the bound

$$|Z_2(\phi, g)| \leqslant C_3 g^{\sigma_2} Z_{QU}(\phi, g) \tag{104}$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_2]$.

The following estimates prove that (104) is preserved by the renormalization group if the constants are suitably puzzled together.

Under these assumptions on Z_1 and Z_2, the modulus of (101) obeys the bound

$$
\begin{aligned}
&|R_{Z_1}(Z_2)(\psi, g)| \\
&\quad \leqslant \int_0^1 \mathrm{d}s \int \mathrm{d}\mu_\gamma(\zeta)\, \alpha\{|Z_1(\beta\psi+\zeta, \delta g)| + s\,|Z_2(\beta\psi+\zeta, \delta g)|\}^{\alpha-1} \\
&\quad\quad \times |Z_2(\beta\psi+\zeta, \delta g)| \\
&\quad \leqslant \int_0^1 \mathrm{d}s\, \alpha\{C_1 + s\, C_3(\delta g)^{\sigma_2}\}^{\alpha-1}\, C_3(\delta g)^{\sigma_2} \int \mathrm{d}\mu_\gamma(\zeta)\, Z_{QU}(\beta\psi+\zeta, \delta g)^\alpha \\
&\quad = \{(C_1 + C_3(\delta g)^{\sigma_2})^\alpha - C_1^\alpha\}\, Z_{QU}(\psi, g) \qquad (105)
\end{aligned}
$$

and thus

$$
|R_{Z_1}(Z_2)(\psi, g)| \leqslant \alpha\delta^{\sigma_2}\{C_1 + C_3(\delta g)^{\sigma_2}\}^{\alpha-1}\, C_3\, g^{\sigma_2}\, Z_{QU}(\psi, g) \qquad (106)
$$

for all $\psi \in \mathbb{R}$ and $g \in [0, \inf\{g_1, g_2\}]$.

To otain an iterating bound, $\alpha\delta^{\sigma_2}$ has to be smaller than $\{C_1 + C_3(\delta g)^{\sigma_2}\}^{\alpha-1}$. The volume factor $\alpha = L^D$ is working against us. The flow factor $\delta^{\sigma_2} = L^{-(4-D)\sigma_2}$ is on our side. Furthermore, we can use the smallness of g.

8.3. Estimates on Z_1 and Z_2

The bound (106) involves the positive constants C_1, C_3, and σ_2. We will adjust them so as to obtain an iterating bound. Start from the following knowledge of Z_1.

Let C_4, σ_3, and g_3 be positive constants. Let Z_1 obey the bound

$$
|Z_1(\phi, g)| \leqslant \mathrm{e}^{C_4 g^{\sigma_3}}\, Z_{QU}(\phi, g) \qquad (107)
$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_3]$.

We will not have to compute optimal values of these constants for any choice of approximate fixed point but can content ourselves with the fact that they exist.

Then there exists a positive constant g_4 such that we have the bound

$$
|Z_1(\phi, g)| \leqslant C_5\, Z_{QU}(\phi, g), \qquad C_5 = 1 + \frac{1}{2(\alpha-1)} \qquad (108)
$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_4]$.

We have assigned C_1 a definite value at the expense of a diminution of g_3. The first bound of (103) follows, with $C_1 = C_5$ and $g_1 = g_4$.

Let C_3 and σ_2 be given. Then there exists a positive constant g_6 such that

$$C_3(\delta g)^{\sigma_2} \leqslant \frac{1}{2(\alpha - 1)} \tag{109}$$

for all $g \in [0, g_6]$.

From (108) and (109) it follows that

$$\{C_1 + C_3(\delta g)^{\sigma_2}\}^{\alpha - 1} \leqslant \left(1 + \frac{1}{\alpha - 1}\right)^{\alpha} \leqslant \mathrm{e} \tag{110}$$

for all $g \in [0, \inf\{g_2, ..., g_6\}]$.

Let α, δ, and σ_2 be such that

$$\alpha\delta^{\sigma_2} \leqslant \frac{1}{2\mathrm{e}} \tag{111}$$

For a given scale L and a given dimension D, with $L > 1$ and $D < 4$, this can be achieved by making σ_2 sufficiently large. For a given dimension D and a given exponent σ_2, with $D < 4$ and $D - (4 - D)\,\sigma_2 < 0$, this can be achieved by making L sufficiently large. We will use the first mechanism to ensure the validity of (111).

Then we have that (106) satisfies the bound

$$|R_{Z_1}(Z_2)(\psi, g)| \overset{(110)}{\leqslant} \alpha\delta^{\sigma_2}\mathrm{e}C_3 g^{\sigma_2} Z_{QU}(\psi, g) \overset{(111)}{\leqslant} \frac{C_3}{2} g^{\sigma_2} Z_{QU}(\psi, g) \tag{112}$$

for all $\psi \in \mathbb{R}$ and $g \in [0, \inf\{g_2, ..., g_6\}]$.

8.4. Estimate on $\Delta(Z_1)$

Assume the following estimate on $\triangle(Z_1)$ from the interpolation technology.

Let C_6, σ_4, and $g_7 > 0$ be positive constants such that we have the bound

$$|\triangle(Z_1)(\psi, g)| \leqslant C_6 g^{\sigma_4} Z_{QU}(\psi, g) \tag{113}$$

for all $\psi \in \mathbb{R}$ and $g \in [0, g_7]$.

We can then use a fraction of g^{σ_4} to diminish C_6.

Then there exists a positive constants C_7, σ_5, and g_8, with $\sigma_5 < \sigma_4$, such that

$$|\triangle(Z_1)(\psi, g)| \leqslant C_7 g^{\sigma_5} Z_{QU}(\psi, g), \qquad C_7 = \frac{C_3}{2} \tag{114}$$

for all $\psi \in \mathbb{R}$ and $g \in [0, g_8]$.

Consequently, the second bound of (103) holds with $C_2 = C_7$ and $g_1 = g_8$. The exponent σ_5 can be chosen arbitrary close to σ_4 at the expense of a further diminuation of g_8. The bounds are then valid only for very small couplings.

With all above assumptions it follows that

$$\begin{aligned} |\triangle(Z_1)(\psi, g) + R_{Z_1}(Z_2)(\psi, g)| &\leqslant |\triangle(Z_1)(\psi, g)| + |R_{Z_1}(Z_2)(\psi, g)| \\ &\leqslant C_3 g^{\sigma_2} Z_{QU}(\psi, g) \end{aligned} \tag{115}$$

for all $\psi \in \mathbb{R}$ and $g \in [0, \inf\{g_2, \ldots, g_8\}]$.

The subspace of functions Z_2, which obey the bound (104), then form an invariant cone around the approximate fixed point Z_1. The program is now to use the contraction mapping principle to prove the existence of a unique element Z_2 such that $Z_1 + Z_2$ is a fixed point of $R \times \delta^\star$. To summarize the logic:

8.5. Iterating Bound

Let $Z_1(\phi, g)$ be a function which satisfies the following stability bound (I) in addition to the following error bound (II).

(I) There exist positive constants C_4, σ_3, and g_3 such that

$$|Z_1(\phi, g)| \leqslant e^{C_4 g^{\sigma_3}} Z_{QU}(\phi, g) \tag{116}$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_3]$.

(II) There exist positive constants C_6, σ_4, and g_7 such that

$$|\triangle(Z_1)| \leqslant C_6 g^{\sigma_4} Z_{QU}(\psi, g) \tag{117}$$

for all $\psi \in \mathbb{R}$ and $g \in [0, g_7]$.

Then we have proved the following.

For all positive constants C_3 and σ_2, with $\sigma_2 < \sigma_4$, there exists a positive constant g_9 such that:

If $Z_2(\phi, g)$ is another function which satisfies the bound

$$|Z_2(\phi, g)| \leqslant C_3 g^{\sigma_2} Z_{QU}(\phi, g) \tag{118}$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_9]$, then $\Delta(Z_1)(\psi, g) + R_{Z_1}(Z_2)(\psi, g)$ is a function in $\mathscr{X}$, which satisfies the bound

$$|\Delta(Z_1)(\psi, g) + R_{Z_1}(Z_2)(\psi, g)| \leqslant C_3 g^{\sigma_2} Z_{QU}(\psi, g) \tag{119}$$

for all $\psi \in \mathbb{R}$ and $g \in [0, g_9]$.

The closer the value of σ_2 is to the value of σ_4, and the larger the value of C_3 is, the smaller is the value of g_9.

We are thus led to search for an approximate fixed point $Z_1(\phi, g)$ which satisfies (I) and (II); in particular (II) with as large σ_4 as possible.

Our notion of an invariant cone is mathematically somewhat casual. To be precise, we should introduce another Banach space of continuous functions Z_2 with respect to the norm[2]

$$\|Z_2\| = \sup_{\phi \in \mathbb{R},\, g \in (0, g_2]} \frac{|Z_2(\phi, g)|}{g^{\sigma_2} Z_{QU}(\phi, g)} \tag{120}$$

We should then consider perturbations Z_2 in a ball around the origin in this new Banach space. The contraction mapping principle applies to this ball. We have an imbedding of this new Banach space into old one. The invariant cone is then really the intersection of an imbedded ball with a ball in $\mathscr{X}$. Because this should not lead to confusion, we continue with the somewhat casual notion of our invariant cone.

9. CONTRACTION MAPPING PRINCIPLE

In the previous section we have constructed an invariant cone of perturbations Z_2 of a given approximate fixed point Z_1. We will now show that the distance of the images of any two perturbations Z_2 and Z'_2 is strictly smaller than the distance of Z_2 and Z'_2. This property is called the contraction mapping property.

9.1. Differential Estimate

Consider two elements Z_2 and Z'_2 in our invariant cone. Then we have that

$$(R \times \delta^\star)(Z_1 + Z_2) - (R \times \delta^\star)(Z_1 + Z'_2) = R_{Z_1}(Z_2) - R_{Z_1}(Z'_2) \tag{121}$$

[2] I am grateful to the referee for bringing this point to my attention.

and

$$R_{Z_1}(Z_2) - R_{Z_1}(Z_2') = \int_0^1 \mathrm{d}s \frac{\partial}{\partial s} R_{Z_1}(Z_2' + s(Z_2 - Z_2'))$$
$$= \int_0^1 \mathrm{d}s \left[\frac{\partial}{\partial s'}\right]_{s'=0} R_{Z_1}(Z_2' + s(Z_2 - Z_2') + s'(Z_2 - Z_2'))$$
$$= \int_0^1 \mathrm{d}s\, D_{Z_2' + s(Z_2 - Z_2')} R_{Z_1}(Z_2 - Z_2') \tag{122}$$

A few words on its justification:

(I) Our invariant cone is convex since, for any two elements Z_2 and Z_2', the linear combination $Z_2' + s(Z_2 - Z_2')$ is an element of it for all $s \in [0, 1]$. The reason is simply that

$$\|Z_2' + s(Z_2 - Z_2')\| \leqslant (1 - s)\, \|Z_2'\| + s\, \|Z_2\| \tag{123}$$

The linear combination $Z_2' + s(Z_2 - Z_2')$ is in particular in the domain of the nonlinear operator R_{Z_1}.

(II) For all values of $s \in (0, 1)$, the function $R_{Z_1}(Z_2' + s(Z_2 - Z_2') + s'(Z_2 - Z_2'))$ is continuously differentiable in s. Therefore we have the identity (122).

Suppose that we can show that for all $s \in [0, 1]$:

$$\|D_{Z_2' + s(Z_2 - Z_2')} R_{Z_1}(Z_2 - Z_2')\| \leqslant \lambda\, \|Z_2 - Z_2'\|, \qquad 0 < \lambda < 1 \tag{124}$$

Then it follows that

$$\|R_{Z_1}(Z_2) - R_{Z_1}(Z_2')\| \leqslant \lambda\, \|Z_2 - Z_2'\| \tag{125}$$

and we have shown that R_{Z_1} and hence $R \times \delta^\star$ is a contraction mapping.

We thus have to show that the operator norm of the linearization of R_{Z_1} satisfies the bound

$$\|D_{Z_2} R_{Z_1}\| = \sup_{\|Z_2'\| = 1} \|D_{Z_2} R_{Z_1}(Z_2')\| \leqslant \lambda, \qquad 0 < \lambda < 1 \tag{126}$$

for all elements Z_2 of the invariant cone. In other words, we have to show that the operator norm of the linearization of R_{Z_1} is strictly smaller than one.

9.2. Operator Norm of $D_{Z_2}R_{Z_1}$

The linearization of R_{Z_1} at Z_2 applied to Z'_2 is given by

$$
\begin{aligned}
&D_{Z_2}R_{Z_1}(Z'_2)(\psi, g)\\
&\quad=\left[\frac{\partial}{\partial\varepsilon}\right]_{\varepsilon=0} R_{Z_1}(Z_2+\varepsilon Z'_2)(\psi, g)\\
&\quad=\int_0^1 \mathrm{d}s\int \mathrm{d}\mu_\gamma(\zeta)\,\alpha\{Z_1(\beta\psi+\zeta,\delta g)+s\,Z_2(\beta\psi+\zeta,\delta g)\}^{\alpha-2}\\
&\qquad\times\{Z_1(\beta\psi+\zeta,\delta g)+\alpha s\,Z_2(\beta\psi+\zeta,\delta g)\}\,Z'_2(\beta\psi+\zeta,\delta g)\\
&\quad=\int \mathrm{d}\mu_\gamma(\zeta)\,\alpha\{Z_1(\beta\psi+\zeta,\delta g)+Z_2(\beta\psi+\zeta,\delta g)\}^{\alpha-1}\,Z'_2(\beta\psi+\zeta,\delta g)
\end{aligned}
\tag{127}
$$

Let us assume that Z_1, Z_2, and Z'_2 satisfy the following bounds.

Let C_1, C_3, C'_3, σ_2, and g_0 be positive constants such that we have the bounds

$$|Z_1(\phi, g)|\leqslant C_1\,Z_{QU}(\phi, g) \tag{128}$$

and

$$|Z_2(\phi, g)|\leqslant C_3\,g^{\sigma_2}\,Z_{QU}(\phi, g),\qquad |Z'_2(\phi, g)|\leqslant C'_3\,g^{\sigma_2}\,Z_{QU}(\phi, g) \tag{129}$$

for all $\phi\in\mathbb{R}$ and $g\in[0, g_0]$.

Then it follows that

$$
\begin{aligned}
&|D_{Z_2}R_{Z_1}(Z'_2)(\psi, g)|\\
&\quad\leqslant\int \mathrm{d}\mu_\gamma(\zeta)\,\alpha\{|Z_1(\beta\psi+\zeta,\delta g)|+|Z_2(\beta\psi+\zeta,\delta g)|\}^{\alpha-1}\,|Z'_2(\beta\psi+\zeta,\delta g)|\\
&\quad\leqslant\alpha\{C_1+C_3(\delta g)^{\sigma_2}\}^{\alpha-1}\,C'_3(\delta g)^{\sigma_2}\int \mathrm{d}\mu_\gamma(\zeta)\,Z_{QU}(\beta\psi+\zeta,\delta g)^\alpha\\
&\quad=\alpha\delta^{\sigma_2}\{C_1+C_3(\delta g)^{\sigma_2}\}^{\alpha-1}\,C'_3\,g^{\sigma_2}\,Z_{QU}(\psi, g)
\end{aligned}
\tag{130}
$$

Under all assumptions made in the previous section, we have that

$$\alpha\delta^{\sigma_2}\{C_1+C_3(\delta g)^{\sigma_2}\}^{\alpha-1}\leqslant\tfrac{1}{2} \tag{131}$$

Divide by C_3' to obtain

$$\|D_{Z_2} R_{Z_1}\| \leqslant \lambda, \qquad \lambda = \tfrac{1}{2} \tag{132}$$

for all Z_2 in the invariant cone.

As a consequence, we have a contraction mapping. The contraction mapping principle implies the existence of a unique fixed point in the invariant cone.

9.3. Discussion

The contraction property relies on a sufficiently close approximation Z_1 to the fixed point. The parameters, which determine how close an approximation is, are the scale L and an exponent σ. For the simplest approximation

$$Z_1(\phi, g) = \mathrm{e}^{-(g/4!)\, P_4(\phi, v)} \tag{133}$$

we found $\sigma = \frac{1}{2}$. The break even dimension for this case, defined by

$$\alpha\delta^{\sigma} = L^{D-(4-D)\sigma} = 1, \qquad D = \frac{4\sigma}{1+\sigma} \tag{134}$$

is in this case $\frac{4}{3}$. To go to higher dimensions, we either have to find a better estimate for the error of the linear approximation, or we have to use a better approximate fixed point by means of higher interpolations.

We should say that we are overweighting the volume factor α by the flow factor δ^{σ}. In the literature on mathematical renormalization theory, the volume factor is usually divided out by taking the normalized renormalization group. In this situation one has to deal with the subleading mass factor L^2, the subleading relevant eigenvalue of the linearized renormalization group. This idea could also be implemented into our fixed point scheme by the imposition of renormalization conditions and Taylor expansions in the field variable.

10. LINEAR APPROXIMATION II

Reconsider the linear approximation to the ϕ^4-trajectory, given by

$$Z(\phi, g) = \mathrm{e}^{-V(\phi, g)}, \qquad V(\phi, g) = \frac{g}{4!} P_4(\phi, v) \tag{135}$$

The previous bound on its error as a fixed point of $R \times \delta$ used a bound on its growth for large imaginary ϕ together with Cauchy estimate. This growth dictated the value $g^{1/4}$ for the Cauchy radius, which then led to a power $g^{1/2}$ in the estimate. To get a better understanding of the factor $g^{1/2}$, we will rederive this bound using only real estimates.

10.1. Large-Field Domination

Let $\phi \in \mathbb{R}$ in the following. Then we have that

$$gP_4(\phi, v) \geqslant \frac{g}{2}\phi^4 - 15gv^2 \tag{136}$$

and

$$\begin{aligned} g^2\left(\frac{\partial}{\partial\phi} P_4(\phi, v)\right)^2 &= g^2(4\phi^3 - 12v\phi)^2 \\ &= 16g^2(\phi^6 - 6v\phi^4 + 9v^2\phi^2) \\ &\leqslant 16g^2(\phi^6 + 9v^2\phi^2) \end{aligned} \tag{137}$$

It follows that

$$\begin{aligned} &\exp\{-gP_4(\phi, v)\}\; g^2\left(\frac{\partial}{\partial\phi} P_4(\phi, v)\right)^2 \\ &\leqslant \exp\left(-\frac{g}{2}\phi^4 + 15gv^2\right) 16g^2(\phi^6 + 9v^2\phi^2) \\ &= \exp\left(-\frac{g}{4}\phi^4 + 15gv^2\right)\exp\left(-\frac{1}{4}(g^{1/4}\phi)^4\right) \\ &\quad \times 16\sqrt{g}\,((g^{1/4}\phi)^6 + 9gv^2(g^{1/4}\phi)^2) \\ &\leqslant \exp\left(-\frac{g}{4}\phi^4 + 15gv^2\right) 16\sqrt{g}\,(C_1 + 9gv^2C_2) \end{aligned} \tag{138}$$

with

$$C_1 = \sup_{\Phi\in\mathbb{R}}\left\{\exp\left(-\frac{\Phi^4}{4}\right)\Phi^6\right\}, \qquad C_2 = \sup_{\Phi\in\mathbb{R}}\left\{\exp\left(-\frac{\Phi^4}{4}\right)\Phi^2\right\} \tag{139}$$

In the estimate (138), we rediscover the factor $\sqrt{g}$. The remaining exponential factor on the right hand side of (139) can be bounded by the quadratic fixed point. Notice that this bound is true for all values of g. We come to the following conclusion.

Let $V(\phi, g) = (g/4!)\, P_4(\phi, v)$. There exist positive constants C_3, C_4, and g_1 such that

$$\exp(-V(\phi, g))\left(\frac{\partial}{\partial\phi} V(\phi, g)\right)^2 \leqslant C_3 \sqrt{g} \exp\left(-\frac{C_4\sqrt{g}}{2}\phi^2\right) \tag{140}$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_1]$.

10.2. Interpolation Formula

Recall the interpolation formula for the linear approximation:

$$(R \times \delta^\star - \mathrm{i}d)(Z)(\psi, g) = \int_0^1 \mathrm{d}s \frac{\partial}{\partial s} R_s(Z)(\psi, g) \tag{141}$$

with

$$\begin{aligned}\frac{\partial}{\partial s} R_s(Z)(\psi, g) & \\ &= \frac{\gamma}{2} \int \mathrm{d}\mu_{s\gamma}(\zeta) \exp\left\{-\alpha \int \mathrm{d}\mu_{(1-s)\gamma}(\zeta)\; V(\beta\psi + \zeta + \xi, \delta g)\right\} \\ &\quad \times \left\{\frac{\partial}{\partial(\beta\psi)} \alpha \int \mathrm{d}\mu_{(1-s)\gamma}(\xi)\; V(\beta\psi + \zeta + \xi, \delta g)\right\}^2\end{aligned} \tag{142}$$

Large field domination implies the following bound.

Let $V(\phi, g) = (g/4!)\, P_4(\phi, v)$. There exist positive constants C_5, C_6, and g_2 such that

$$\begin{aligned}&\exp\left\{-\alpha \int \mathrm{d}\mu_{(1-s)\gamma}(\xi)\; V(\phi + \xi, \delta g)\right\}\left\{\frac{\partial}{\partial\phi} \alpha \int \mathrm{d}\mu_{(1-s)\gamma}(\xi)\; V(\phi + \xi, \delta g)\right\}^2 \\ &\quad \leqslant C_5 \sqrt{g} \exp\left(-\frac{C_6\sqrt{g}}{2}\phi^2\right)\end{aligned} \tag{143}$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_2]$.

It follows that

$$
\begin{aligned}
&\frac{\partial}{\partial s} R_s(Z)(\psi, g) \\
&\quad \leqslant \frac{\gamma C_5 \sqrt{g}}{2} \int d\mu_{s\gamma}(\zeta) \exp\left(-\frac{C_6 \sqrt{g}}{2} (\beta\psi + \zeta)^2\right) \\
&\quad = \frac{\gamma C_5 \sqrt{g}}{2} (1 + s\gamma C_6 \sqrt{g})^{-1/2} \exp\left\{-\frac{C_6 \sqrt{g}\, \beta^2}{2} (1 + s\gamma C_6 \sqrt{g})^{-1} \psi^2\right\} \\
&\quad \leqslant \frac{\gamma C_5 \sqrt{g}}{2} \exp\left\{-\frac{C_6 \sqrt{g}\, \beta^2}{2} (1 + \gamma C_6 \sqrt{g})^{-1} \psi^2\right\} \qquad (144)
\end{aligned}
$$

The next step is to estimate the exponential by the quadratic fixed point. For sufficiently small g this is possible in dimensions $D > 0$, which we will assume subsequently. The error term of the linear approximation therefore satisfies the following bound.

There exist positive constants C_7, C_8, σ and g_3 such that

$$(R \times \delta^\star - \mathrm{i}d)(Z)(\psi, g) \leqslant C_7 \sqrt{g} \exp(C_8 g^\sigma)\, Z_{QU}(\psi, g) \qquad (145)$$

for all $\psi \in \mathbb{R}$ and $g \in [0, g_3]$.

Put $C_9 = C_7 e^{C_8 g^{\sigma_3}}$ to obtain

$$(R \times \delta^\star - \mathrm{i}d)(Z)(\psi, g) \leqslant C_9 \sqrt{g}\, Z_{QU}(\psi, g) \qquad (146)$$

We have therefore indeed and an invariant cone around the linear approximation. The power $g^{1/2}$ is however not sufficient to imply a contraction mapping at dimensions close to three.

11. STABILITY AT HIGHER ORDERS

Perturbation theory supplies us with a sequence of polynomial approximants for the ϕ^4-trajectory. We will prove a large field bound for the higher order perturbative potentials. The bound requires explicit knowledge of the perturbative potential in the tree approximation. We will prove it for the orders three, five, and seven.

11.1. Perturbative Potential

Recall that the hierarchical renormalization group transformation (4) has two parameters, a scale L and a dimension D. In our perturbative calculation, we trade the dimension D for the volume $\alpha = L^D$. We thus choose the parameter values

$$\beta = \frac{L}{\sqrt{\alpha}}, \qquad \gamma = 1 - \beta^2, \qquad v = 1 \tag{147}$$

For computational convenience, we normalize the expansion parameter g differently in this section. We choose the normalization such that $V(\phi, g) = gP_4(\phi, v) + O(g_2)$. Furthermore, we put $L = 2$ to reduce the size of our expressions.

The perturbative potential of order r comes out as a polynomial of order $2(r+1)$ in ϕ. It has the general form

$$V(\phi, g) = \sum_{n=0}^{r+1} g^{n-1} \lambda_{2n}(g)\, \phi^{2n} \tag{148}$$

with polynomial coefficients

$$\lambda_{2n}(g) = \sum_{s=0}^{r-n+1} g^s \lambda_{2n}^{(s+n-1)} \tag{149}$$

where $\lambda_0^{(-1)} = 0$. The perturbative coefficients $\lambda_{2n}^{(r)}$ with $n > r+1$ come out as zero, those with $n \leqslant r+1$ come out as rational functions in α. Up to order three they are listed in Table IV. The highest coefficients

$$\lambda_{2n}^{(n-1)} = \lambda_{2n}(0) \tag{150}$$

are called tree graph coefficients. They can be computed independently since their recursion relation decouples from that of the other coefficients. Putting all others to zero defines a tree graph approximation. Up to order seven, they are listed in Table V.

11.2. Tree Graph Bound

For $\alpha > 4$, the tree graph coefficients up to order seven have alternating signs

$$\lambda_{2n}(0) = (-1)^n\, |\lambda_{2n}(0)| \tag{151}$$

as can be seen from Table V. This is also true for all higher orders as can be seen from the tree graph recursion relation, not to be written here. The even order tree graph potentials are therefore unstable. For this reason, we choose the order r to be odd.

Table IV. Perturbative Fixed Point

$\lambda_4^{(1)} = 1$

$\lambda_2^{(1)} = -6$

$\lambda_0^{(1)} = 3$

$\lambda_6^{(2)} = -\frac{8}{3}(\alpha - 4)$

$\lambda_4^{(2)} = 4(\alpha - 4)(19\alpha - 124)(\alpha - 16)^{-1}$

$\lambda_2^{(2)} = -96(\alpha - 4)(3\alpha^3 - 5\alpha^2 - 200\alpha + 832)(\alpha - 16)^{-1}(\alpha - 8)^{-1}(\alpha + 8)^{-1}$

$$\lambda_0^{(2)} = 16(\alpha - 4)^2(7\alpha^5 + 42\alpha^4 - 364\alpha^3 - 1024\alpha^2 - 8448\alpha + 90112)(\alpha - 16)^{-1}(\alpha - 8)^{-1} \times (\alpha + 8)^{-1}(\alpha^3 - 256)^{-1}$$

$\lambda_8^{(3)} = \frac{32}{3}(\alpha - 4)^2$

$\lambda_6^{(3)} = -\frac{640}{3}(\alpha - 4)^2(5\alpha^2 - 193\alpha + 944)(\alpha - 16)^{-1}(\alpha - 64)^{-1}$

$$\lambda_4^{(3)} = 16(\alpha - 4)^2(807\alpha^5 - 20672\alpha^5 - 418512\alpha^4 + 7034624\alpha^3 + 7758848\alpha^2 - 368689152\alpha + 1108344832)(\alpha - 16)^{-2}(\alpha - 64)^{-1}(\alpha - 8)^{-1}(\alpha + 8)^{-1}(\alpha + 16)^{-1}$$

$$\lambda_2^{(3)} = -32(\alpha - 4)^2(1037\alpha^9 - 15436\alpha^8 - 856880\alpha^7 + 5738496\alpha^6 + 32672768\alpha^5 + 448749568\alpha^4 - 4490133504\alpha^3 - 39599472640\alpha^2 + 402971951104\alpha - 871878361088) \times (\alpha - 16)^{-2}(\alpha - 8)^{-1}(\alpha + 8)^{-1}(\alpha - 64)^{-1}(\alpha + 16)^{-1}(\alpha^3 - 1024)^{-1}$$

$$\lambda_0^{(3)} = 16(\alpha - 4)^3(619\alpha^{10} - 1108\alpha^9 - 648736\alpha^8 - 34944\alpha^7 - 41459712\alpha^6 + 440483840\alpha^5 + 1512701952\alpha^4 - 9978249216\alpha^3 + 171815469056\alpha^2 - 1942667395072\alpha + 5480378269696)(\alpha - 8)^{-1}(\alpha + 8)^{-1}(\alpha^2 + 64)^{-1} \times (\alpha^3 - 1024)^{-1} \times (\alpha + 16)^{-1}(\alpha - 64)^{-1}(\alpha - 16)^{-2}$$

Table V. Tree Graph Coefficients

$\lambda_6^{(2)} = -8/3(\alpha - 4)$

$\lambda_8^{(3)} = \frac{32}{3}(\alpha - 4)^2$

$\lambda_{10}^{(4)} = -\frac{1408}{27}(\alpha - 4)^3$

$\lambda_{12}^{(5)} = \frac{23296}{81}(\alpha - 4)^4$

$\lambda_{14}^{(6)} = -\frac{139264}{81}(\alpha - 4)^5$

$\lambda_{16}^{(7)} = \frac{2646016}{243}(\alpha - 4)^6$

The stability bound is constructed recursively. Consider first the order three approximation. Its potential can be estimated by means of

$$\begin{aligned}
&g^2\lambda_6(g)\,\phi^6+g^3\lambda_8(g)\,\phi^8\\
&\quad=g\left(g\frac{\lambda_6(g)}{\lambda_8(g)}\phi^2+g^2\phi^4\right)\lambda_8(g)\,\phi^4\\
&\quad=g\left\{\left(\frac{\lambda_6(g)}{2\lambda_8(g)}+g\phi^2\right)^2-\left(\frac{\lambda_6(g)}{2\lambda_8(g)}\right)^2\right\}\lambda_8(g)\,\phi^4\\
&\quad\geqslant -g\frac{\lambda_6(g)^2}{4\lambda_8(g)}\phi^4
\end{aligned}\tag{152}$$

It follows that

$$g\lambda_4(g)\,\phi^4+g^2\lambda_6(g)\,\phi^6+g^3\lambda_8(g)\,\phi^8\geqslant g\left(\lambda_4(g)-\frac{\lambda_6(g)^2}{4\lambda_8(g)}\right)\phi^4 \tag{153}$$

The estimate yields an effective ϕ^4-coupling constant

$$\rho_4(g)=\lambda_4(g)-\frac{\lambda_6(g)^2}{4\lambda_8(g)} \tag{154}$$

Its value at $g=0$, i.e., its tree graph value, is computed to

$$\rho_4(0)=\tfrac{5}{6} \tag{155}$$

independent of α. Since (154) is a rational function of g, it is continuous. Thus positivity of (155) at $g=0$ also holds for small g. We come to the following conclusion. Fix the dimension D, avoiding resonances.

There exists a positive constant $g_0>0$ such that

$$\rho_4(g)\geqslant\tfrac{1}{2} \tag{156}$$

for all $g\in[0,\, g_0]$.

To third order of perturbation theory, the potential satisfies the lower bound

$$\sum_{n=0}^{4} g^{n-1}\lambda_{2n}(g)\,\phi^{2n}\geqslant g^{-1}\lambda_0(g)+\lambda_2(g)\,\phi^2+g\rho_4(g)\,\phi^4 \tag{157}$$

with polynomial coefficients λ_0 and λ_2 and rational coefficient $\rho_4(g)$. For small g, we have a stability bound as in the linear approximation.

This scheme iterates immediately to any odd order of perturbation theory. To fifth order of perturbation theory we first estimate

$$g^4\lambda_{10}(g)\,\phi^{10}+g^5\lambda_{12}(g)\,\phi^{12}\geqslant -g^3\,\frac{\lambda_{10}(g)^2}{4\lambda_{12}(g)}\,\phi^8 \tag{158}$$

to obtain an effective ϕ^8-coupling

$$\rho_8(g)=\lambda_8(g)-\frac{\lambda_{10}(g)^2}{4\lambda_{12}(g)} \tag{159}$$

We then estimate as above

$$g^2\lambda_6(g)\,\phi^6+g^3\rho_8(g)\,\phi^8\geqslant -g\,\frac{\lambda_6(g)^2}{4\rho_8(g)}\,\phi^4 \tag{160}$$

to obtain an effective ϕ^4-coupling

$$\rho_4(g)=\lambda_4(g)-\frac{\lambda_6(g)^2}{4\rho_8(g)}=\lambda_4(g)-\frac{\lambda_6(g)^2}{4\lambda_8(g)-\lambda_{10}(g)^2/\lambda_{12}(g)} \tag{161}$$

Its tree graph value comes out as

$$\rho_4(0)=\tfrac{334}{425} \tag{162}$$

Therefore, we have again stability for small g.

The seventh order approximant is estimated by three bounds along these lines. The tree graph value of the effective ϕ^4-coupling to seventh order of perturbation theory is

$$\rho_4(0)=\tfrac{4306}{5627} \tag{163}$$

Thus also the seventh order approximant is stable for small g. It follows that we have the following stability bound for the orders one, three, five, and seven.

There exist positive constants C_1, σ_1, and g_1 such that $Z(\phi, g)=e^{-V(\phi,g)}$ satisfies the bound

$$|Z(\phi, g)|\leqslant e^{C_1 g^{\sigma_1}}\,Z_{QU}(\phi, g) \tag{164}$$

for all $\phi\in\mathbb{R}$ and $g\in[0, g_1]$.

Presumably, this bound holds to all orders of perturbation theory with g_1 shrinking to zero as the order tends to infinity.

12. INTERPOLATED CUMULANTS

In this section we derive an identity for interpolated cumululants using a differential equation for the potential. The identity will be needed in the next section.

12.1. Differential Equation

The interpolated Gaussian convolution

$$Z_s(\psi) = \int d\mu_{(1-s)\gamma}(\zeta)\, Z_0(\psi+\zeta) \tag{165}$$

satisfies the differential equation

$$\frac{\partial}{\partial s} Z_s(\psi) = -\frac{\gamma}{2}\frac{\partial^2}{\partial\psi^2} Z_s(\psi) \tag{166}$$

It follows that

$$V_s(\psi) = -\log Z_s(\psi) \tag{167}$$

satisfies the differential equation

$$\frac{\partial}{\partial s} V_s(\psi) = \frac{\gamma}{2}\left\{\left(\frac{\partial}{\partial\psi} V_s(\psi)\right)^2 - \frac{\partial^2}{\partial\psi^2} V_s(\psi)\right\} \tag{168}$$

12.2. Cumulant Expansion

Let Z_0 be given by

$$Z_0(\phi, g) = e^{-V_0(\phi, g)}, \qquad V_0(\phi, g) = \sum_{r=1}^{\infty} V_0^{(r)} \frac{g^r}{r!} \tag{169}$$

Then we have that

$$V_s(\psi, g) = \sum_{r=1}^{\infty} V_s^{(r)}(\psi) \frac{g^r}{r!} \tag{170}$$

with

$$V_s^{(r)}(\psi) = \sum_{i=1}^{r} \frac{(-1)^{i+1}}{i!} \sum_{\sum_{j=1}^{i} s_j = r} \frac{r!}{\prod_{j=1}^{i} s_j!} \left\langle \prod_{j=1}^{i} [V_0^{(s_j)};] \right\rangle^{T}_{(1-s)\gamma,\psi} \tag{171}$$

Comparing equal orders of g we get

$$\left(\frac{\partial}{\partial s}+\frac{\gamma}{2}\frac{\partial^2}{\partial\psi^2}\right) V_s^{(r)}(\psi)=\sum_{t=1}^{r-1}\binom{r}{t}\frac{\gamma}{2}\frac{\partial}{\partial\psi} V_s^{(t)}(\psi)\frac{\partial}{\partial\psi} V_s^{(r-t)}(\psi) \tag{172}$$

and thus

$$\begin{aligned}
&\left(\frac{\partial}{\partial s}+\frac{\gamma}{2}\frac{\partial^2}{\partial\psi^2}\right)\sum_{r=1}^{u} V_s^{(r)}(\psi)\frac{g^r}{r!}\\
&=\sum_{r=1}^{u}\frac{g^r}{r!}\sum_{t=1}^{r-1}\binom{r}{t}\frac{\gamma}{2}\frac{\partial}{\partial\psi} V_s^{(t)}(\psi)\frac{\partial}{\partial\psi} V_s^{(r-t)}(\psi)\\
&=\frac{\gamma}{2}\left(\frac{\partial}{\partial\psi}\sum_{r=1}^{u} V_s^{(r)}(\psi)\frac{g^r}{r!}\right)^2-\frac{1}{u!}\int_0^1 dt(1-t)^u\\
&\quad\times\frac{\partial^{u+1}}{\partial t^{u+1}}\frac{\gamma}{2}\left(\frac{\partial}{\partial\psi}\sum_{r=1}^{u} V_s^{(r)}(\psi)\frac{(tg)^r}{r!}\right)^2
\end{aligned} \tag{173}$$

Equation (173) is an identity for polynomial functions, which is all we need here.

13. BEYOND THE LINEAR APPROXIMATION

Perturbation theory yields a sequence of polynomial approximants

$$V(\phi, g)=\sum_{s=1}^{r} V^{(s)}(\phi)\frac{g^s}{s!},\qquad V^{(s)}(\phi)=\sum_{n=0}^{s+1}\frac{V_{2n}^{(s)}}{(2n)!}P_{2n}(\phi, v) \tag{174}$$

for the ϕ^4-trajectory. The polynomials $V^{(s)}(\phi)$ are determined such that for all $s\in\{1,\ldots, r\}$, we have the perturbative so called scaling relations

$$V^{(s)}(\psi)=\sum_{i=1}^{s}\frac{(-1)^{i+1}}{i!}\sum_{\sum_{j=1}^{i} s_j=s}\frac{s!}{\prod_{j=1}^{i} s_j!}\left\langle\prod_{j=1}^{i}\left[\alpha V^{(s_j)};\right]\right\rangle_{\gamma,\beta\psi}^{T}\delta^s \tag{175}$$

In this section, we investigate the nonperturbative error up to which these perturbative approximants solve the fixed point equation for a given order $r\in 2\mathbb{N}+1$.

13.1. Interpolated Cumulants

Consider the following interpolation of the cumulant expansion to order r. Expand

$$T_t(V)(\psi, g) = \sum_{s=1}^{r} T_t(V)^{(s)}(\psi)\frac{g^s}{s!} \tag{176}$$

with coefficients

$$T_t(V)^{(s)}(\psi) = \sum_{i=1}^{s} \frac{(-1)^{i+1}}{i!} \sum_{\sum_{j=1}^{i} s_j = s} \frac{s!}{\prod_{j=1}^{i} s_j!} \left\langle \prod_{j=1}^{i} [V^{(s_j)};] \right\rangle^{T}_{(1-t)\gamma,\psi} \tag{177}$$

The interpolation parameter t goes once more from zero to one. Supplemented with scale factors, it interpolates between

$$T_0(\alpha V)(\beta\psi, \delta g) = V(\psi, g), \qquad T_1(\alpha V)(\beta\psi, \delta g) = \alpha V(\beta\psi, \delta g) \tag{178}$$

All operations are well defined since we are working with a polynomial potential.

13.2. Interpolated Renormalization Group

Consider then the following interpolated renormalization group. Let

$$R_t(Z)(\psi, g) = \int d\mu_{t\gamma}(\zeta) \exp(-T_t(\alpha V)(\beta\psi + \zeta, \delta g)) \tag{179}$$

with t going from zero to one. It interpolates between

$$R_0(Z)(\psi, g) = \exp(-T_0(\alpha V)(\beta\psi, \delta g)) = \exp(-V(\psi, g)) = Z(\psi, g) \tag{180}$$

and

$$R_1(Z)(\psi, g) = \int d\mu_\gamma(\zeta) \exp(-\alpha V(\beta\psi + \zeta, \delta g)) = (R \times \delta^\star)(Z)(\psi, g) \tag{181}$$

It can thus be used to estimate the error up to which Z is a fixed point of $R \times \delta^\star$. We have that

$$(R \times \delta^\star - id)(Z)(\psi, g) = \int_0^1 dt \frac{\partial}{\partial t} R_t(Z)(\psi, g) \tag{182}$$

where

$$\frac{\partial}{\partial t} R_t(Z)(\psi, g)$$
$$= \int d\mu_{t\gamma}(\zeta) \exp(-T_t(\alpha V)(\beta\psi + \zeta, \delta g))$$
$$\times \left\{ -\left(\frac{\partial}{\partial t} + \frac{\gamma}{2} \frac{\partial^2}{\partial(\beta\psi)^2}\right) T_t(\alpha V)(\beta\psi + \zeta, \delta g) + \frac{\gamma}{2}\left(\frac{\partial}{\partial(\beta\psi)} T_t(\alpha V)(\beta\psi + \zeta, \delta g)\right)^2 \right\} \tag{183}$$

From the interpolation formula for cumulants we obtain

$$\frac{\partial}{\partial t} R_t(Z)(\psi, g)$$
$$= \int d\mu_{t\gamma}(\zeta) \exp(-T_t(\alpha V)(\beta\psi + \zeta, \delta g))$$
$$\times \frac{1}{r!} \int_0^1 du(1-u)^r \frac{\partial^{r+1}}{\partial u^{r+1}} \frac{\gamma}{2}\left(\frac{\partial}{\partial(\beta\psi)} T_t(\alpha V)(\beta\psi + \zeta, u\,\delta g)\right)^2 \tag{184}$$

When $r = 1$, (184) reduces to the previous interpolation formula given by (71). The point with this higher interpolation is that the right hand side of (184) is of the order g^{r+1}. It offers the possibility to obtain a bound with a higher power of g.

13.3. Estimate of the Downstairs Factor

Let us first consider the case when $t = 0$. We look for an estimate of the modulus of the expression

$$\frac{\gamma}{2} e^{-V(\phi, g)} \frac{1}{r!} \int_0^1 du(1-u)^r \frac{\partial^{r+1}}{\partial u^{r+1}} \left\{\frac{\partial}{\partial\phi} V(\phi, ug)\right\}^2 \tag{185}$$

Two facts come handy for this estimate. The first fact is the stability bound on V. The second fact is the order g^{r+1} of the downstairs factor.

There exist positive constants C_1, C_2, σ_1, and g_1 such that

$$|e^{-V(\phi, g)}| \leqslant e^{C_1 g^{\sigma_1}} e^{-C_2 g \phi^4} \tag{186}$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_1]$.

The perturbative potential is a polynomial of the form

$$V(\phi, g) = \sum_{n=0}^{r+1} g^{n-1} \lambda_{2n}(g)\, \phi^{2n} \tag{187}$$

where $\lambda_0(g) = O(g)$. It follows that the downstairs factor is a polynomial of this same form, namely

$$\begin{aligned} &\frac{1}{r!} \int_0^1 \mathrm{d}u (1-u)^r \frac{\partial^{r+1}}{\partial u^{r+1}} \left\{ \frac{\partial}{\partial \phi} V(\phi, ug) \right\}^2 \\ &\quad = \sum_{n=1}^{r+1} g^{r+1} \mu_{2n}(g)\, \phi^{2n} + \sum_{n=r+2}^{2r+1} g^{n-1} \mu_{2n}(g)\, \phi^{2n} \end{aligned} \tag{188}$$

with certain polynomials $\mu_{2n}(g)$, $n \in \{1,..., 2r+1\}$. The cancellation effects only terms up to the order g^r. The two terms in Eq. (188) are of the orders

$$g^{r+1} \phi^{2n} = g^{r+1-n/2} (g^{1/4} \phi)^{2n}, \qquad r+1-\frac{n}{2} \overset{n<r+2}{\geqslant} \frac{r+1}{2} \tag{189}$$

and

$$g^{n-1} \phi^{2n} = g^{n/2-1} (g^{1/4} \phi)^{2n}, \qquad \frac{n}{2} - 1 \overset{n>r+1}{\geqslant} \frac{r}{2} \tag{190}$$

From this it follows that

$$\begin{aligned} &\left| \mathrm{e}^{-V(\phi, g)} \frac{1}{r!} \int \mathrm{d}t (1-t)^r \frac{\partial^{r+1}}{\partial t^{r+1}} \left\{ \frac{\partial}{\partial \phi} V(\phi, g) \right\}^2 \right| \\ &\quad \leqslant \mathrm{e}^{C_1 g^{a_1}} g^{r/2} \mathrm{e}^{-(C_2/2)(g^{1/4}\phi)^4} \left\{ \sum_{n=1}^{r+1} g^{(r-n)/2+1} |\mu_{2n}(g)| \, (g^{1/4}\phi)^{2n} \right. \\ &\qquad \left. + \sum_{n=r+2}^{2r+1} g^{(n-r)/2-1} |\mu_{2n}(g)| \, (g^{1/4}\phi)^{2n} \right\} \mathrm{e}^{-(C_2/2)\, g\phi^4} \\ &\quad \leqslant \mathrm{e}^{C_1 g^{a_1}} g^{r/2} M(g^{1/2})\, \mathrm{e}^{-(C_2/2)\, g\phi^4} \end{aligned} \tag{191}$$

with a certain polynomial M. Notice that order $g^{7/2}$ in front of this estimate. For the seventh order approximant, we get a power $g^{r/2}$ which is cabable of overweighting even the volume factor above three dimensions. The exponential factor on the right hand side of (191) can then be estimated by the quadratic g-dependent fixed point.

There exist positive constants C_3, C_4, σ_2, and g_2, with $\sigma_2 < r/2$, such that the modulus of (185) is thus bounded by

$$e^{C_3 g^{\sigma_1}} g^{\sigma_2} e^{-(C_4 \sqrt{g}/2)\,\phi^2} \tag{192}$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_2]$.

This bound is valid for $t = 0$. It extends to an analogous bound which holds uniform in $t \in [0, 1]$.

13.4. Error Bound

For all $t \in [0, 1]$, the interpolation $T_t(V)(\phi, g)$ is a polynomial of the same form as $V(\phi, g)$, namely

$$T_t(V)(\phi, g) = \sum_{n=0}^{r+1} g^{n-1} \lambda_{2n}(g, t)\, \phi^{2n} \tag{193}$$

where $\lambda_{2n}(g, t)$ are polynomials in g and t of the form $\lambda_{2n}(g, t) = \lambda_{2n}^{tree}(1 + 3t/\alpha)^{n-2} + gO(g, t))$. We have computed both $V(\phi, g)$ and its interpolation $T_t(V)(\phi, g)$ by means of computer algebra. Since they are lengthy expressions, we will not reproduce them down here. The previous tree graph bound is repeated to give the following result. The effective tree graph coupling $\rho_4(0, t)$, defined as in the case $t = 0$, comes out independent of t. By uniform continuity we find a stability bound which is uniform in the interpolation parameter $t \in [0, 1]$.

There exist positive constants C_5, C_6, σ_3, and g_3, with $\sigma_3 < r/2$, such that

$$\begin{aligned} &\left| e^{-T_t(V)(\phi, g)} \frac{1}{r!} \int_0^1 ds (1-s)^r \frac{\partial^{r+1}}{\partial s^{r+1}} \left\{ \frac{\partial}{\partial \phi} T_t(V)(\phi, sg) \right\}^2 \right| \\ &\quad \leqslant C_5 g^{\sigma_3} \exp\left(\frac{C_6 \sqrt{g}}{2} \phi^2 \right) \end{aligned} \tag{194}$$

for all $\phi \in \mathbb{R}$, $g \in [0, g_3]$, and $t \in [0, 1]$.

As a consequence, we have the following bound.

There exist positive constants C_7, σ_3, and g_4, with $\sigma_3 < r/2$, such that

$$\left| \frac{\partial}{\partial t} R_t(Z)(\psi, g) \right| \leqslant C_7 g^{\sigma_3} Z_{QU}(\psi, g) \tag{195}$$

for all $\phi \in \mathbb{R}$, $g \in [0, g_4]$, and $t \in [0, 1]$.

Therefrom it follows that

$$|(R \times \delta^{\star} - \mathrm{i}d)(Z)(\psi, g)| \leqslant C_7 g^{\sigma_3} Z_{QU}(\phi, g) \tag{196}$$

for all $\phi \in \mathbb{R}$ and $g \in [0, g_4]$.

With this error bound we can go back to the section on the invariant cone and the section on the contraction mapping principle to conclude the existence of the ϕ^4-curve. The important point is that (196) is of the order g^{σ_3} uniform in ϕ, divided by the quadratic fixed point. The higher order approximant is thus close to the true fixed point for small g. The value of σ_3 can be arbitrary close to $r/2$.

The break even dimension for seventh order approximant is

$$D = \frac{4\sigma}{1+\sigma} = \frac{28}{9}, \qquad \sigma = \frac{7}{2} \tag{197}$$

Higher dimensions than this require a higher order perturbation theory and in particular an investigation of its stability. The properties of the perturbative approximants to any order of perturbation theory will not treated here.

13.5. Concluding Remarks

13.5.1. Positivity of $Z(\phi, g)$. We have constructed $Z(\phi, g)$ by means of a contraction mapping in a Banach space (which includes non-positive functions): $Z(\phi, g) = Z_1(\phi, g) + \lim_{n \to \infty} Z_{2,n}(\phi, g)$, where $Z_{2,0}(\phi, g) \mapsto Z_{2,1}(\phi, g) \mapsto Z_{2,2}(\phi, g) \mapsto \cdots \mapsto Z_{2,\infty}(\phi, g)$ is the sequence generated from $Z_{2,0}(\phi, g) = 0$ by the contraction mapping. Since (4) maps positive real valued functions to positive real valued functions, and since $Z_1(\phi, g)$ is positive real valued, the sequence as well as its limit is positive real valued. From this reasoning, it follows that the fixed point $Z(\phi, g)$ is non-negative. By direct inspection, it can be shown that for all $g \in (0, g_2]$ there exists $\Phi(g) > 0$ such that for all $|\phi| < \Phi(g)$ we have that $Z(\phi, g) > 0$. Thus there is a region of small fields, where strict positivity holds. For $|\phi| \geqslant \Phi(g)$, we know that $Z(\phi, g) \geqslant 0$. Suppose that there existed a ϕ_0 with $|\phi_0| \geqslant \Phi(g)$ such that $Z(\phi_0, g) = 0$. Then we would have that

$$\int d\mu_\gamma(\zeta)\, Z(\beta\phi_0 + \zeta \mid \delta g)^\alpha = Z(\phi_0, g) = 0 \tag{198}$$

But the image of the Gaussian convolution is strictly positive. We have a contradiction. We will not make these statements mathematically more precise here but emphasize that the matter of positivity of $Z(\phi, g)$ should be adressed after its construction, which does not use positivity. In particular, there remains the issue of an accurate lower bound on $Z(\phi, g)$ at large ϕ. Although this is an important piece of information, we do not need it in the present construction.

13.5.2. $D=3$ Dimensions. In its present form, the construction excludes $D=3$ dimensions. However, the obstacle is merely technical, not conceptual. In $D=3$ dimensions, $Z_1(\phi, g)=e^{-V_1(\phi, g)}$ does not have a formal power series expansion in g. This can be seen directly from our perturbative fixed point listed in the appendix. The coefficients are rational functions in α and some of them have poles at $\alpha=8$ (or $D=3$ when $L=2$). On the other hand, we know from [RW96] that we can compute perturbative approximants by double expansion in g and $g^2\log(g)$. It turns out that these approximants have all properties needed for the contraction mapping method to apply exactly as above. The point is that the tree graph bound applies exactly as above because the $g^2\log(g)$-insertions are subleading corrections. We postpone a constructive version of the logarithmic singularities on the three dimensional ϕ^4-trajectory to future work.

13.5.3. Dimensions $28/9<D<4$. The seventh order approximant suffices to obtain a contraction mapping in $D<28/9$ dimensions. The order r approximant suffices to obtain a contraction mapping in $D<4r/2+r$ dimensions. This bound is obtained from $D=4\sigma/(1+\sigma)$, see (134), in conjunction with $\sigma=r/2$. We have not proved the stability estimate to all orders r of perturbation theory in this paper.[3] However, such an analysis can be done. It requires a computation of tree graph coefficients and an analysis of the effective couplings in our tree graph estimate. Another problem is that of further resonances. From [RW96], we know that there is a discrete sequence of dimensions other than three where resonances occur. At these dimensions, we cannot use directly the perturbative approximant. However, it is very likely that they can be treated analogous to the three dimensional case outlined above.

14. OUTLOOK

The study of hierarchical models as a laboratory for the renormalization group analysis of asymptotically free full models was advocated in [GK82, GK83, GK86]. Our main motivation for this study is to prepare

[3] We have in fact proved the stability bound up to orders $r=99$ by means of computer algebra.

the ground for an analogous study of the ϕ^4-trajectory in the full setting. From the present construction we anticipate that the perturbative approximants are promising candidates for approximate fixed points also in the full model. Using a momentum space renormalization group, they have been computed in the full model in three and four dimensions in [Wie97]. The perturbative part is indeed very similar to the hierarchical model. We hope to generalize this method to the construction of the massless ϕ^4-trajectory in the three dimensional full model in future work.

The constructive work on the full ϕ^4-theory includes [GJ73, GJ87, FO76, MS77, BCG+80, Bal83, GK86, Gal85, GN85, BG95, P90, Bry92, BDH93, R91], and references therein. A lot of knowledge has thus been gathered on the renormalization of ϕ^4-theory. None of these authors however present the problem in this dynamical systems setting suggested in [Wil70, Wil71, WK74]. We hope that our presentation as a generalized renormalization group fixed point is a natural formulation from the dynamical systems point of view and find that it deserves to be developped further.

Concerning hierarchical models themselves, even more knowledge is available. We mention [CE77, EW86, KW86a, KW86b, KW88a, KW88b, KW91, KW94, GK83, GK86, P90, Alb91]. There remain a number of conceivable improvements of our construction for the hierarchical models. It is conceivable that our fixed point technique can be brought to a form, where one has estimates which are uniform in the coupling g. The reason for my optimism is that the large g limit of our setting yields exactly the framework in which the non-trivial fixed point was constructed [KW86a, KW86b, KW88a, KW88b, KW91, KW94]. It would also be very pleasing to prove that our ϕ^4-trajectory, say in three dimensions, connects the trivial with the non-trivial fixed point, the former serving as ultraviolet fixed point, the latter serving as infrared fixed point. It would finally be very pleasing to prove that the ϕ^4-trajectory intersects the unstable manifold of the non-trivial fixed point transversally. These are global aspects of the renormalization group for which our small coupling analysis does not suffice in its present form. A less sophisticated problem is the divergence of perturbation theory. Using the perturbative approximants to any order of perturbation theory should lead to troubles in the estimates without a large field vs. small field separation. A clear understanding of this mechanism would round up the picture. Last not least, $\log(g)$ corrections deserve a better understanding in three dimensions.

15. APPENDIX

This appendix contains a proof of thew complex stability bound (73) and the real stability bound (136).

15.1. Basic Inequalities

Let a and b be two real numbers. A basic inequality is

$$\left(\varepsilon a - \frac{b}{\varepsilon}\right)^2 = \varepsilon^2 a^2 - 2ab + \frac{b^2}{\varepsilon^2} \geqslant 0, \qquad \varepsilon^2 a^2 + \frac{b^2}{\varepsilon^2} \geqslant 2ab \tag{199}$$

It follows that

$$(a+b)^2 \geqslant a^2 - 2\,|a|\,|b| + b^2 \geqslant (1-\varepsilon^2)\,a^2 + \left(1 - \frac{1}{\varepsilon^2}\right) b^2 \tag{200}$$

For $\varepsilon^2 = \frac{1}{2}$, it follows in particular that

$$(a+b)^2 \geqslant \frac{a^2}{2} - b^2 \tag{201}$$

Another elementary inequality is

$$a^2 \geqslant 2ba - b^2 \tag{202}$$

15.2. Complex Stability Bound

Recall $P_4(\phi, v)$ from Table I. Let $\phi \in \mathbb{R}$ and $\chi \in \mathbb{C}$. Then

$$\Re\{P_4(\phi+\chi, v)\} = \Re\{(\phi+\chi)^4 - 6v(\phi+\chi)^2 + 3v^2\} \tag{203}$$

We have that

$$\begin{aligned} \Re\{(\phi+\chi)^4\} &= (\phi + \Re\chi)^4 - 6(\phi+\Re\chi)^2\,(\Im\chi)^2 + (\Im\chi)^4 \\ &= \{(\phi+\Re\chi)^2 - 3(\Im\chi)^2\}^2 - 8(\Im\chi)^2 \\ &\overset{(201)}{\geqslant} \frac{1}{2}(\phi+\Re\chi)^4 - 17(\Im\chi)^4 \\ &\overset{(202)}{\geqslant} b(\phi+\Re\chi)^2 - 17(\Im\chi)^4 - \frac{b^2}{2} \end{aligned} \tag{204}$$

and

$$\Re\{(\phi+\chi)^2\} = (\phi+\Re\chi)^2 - (\Im\chi)^2 \tag{205}$$

It follows that

$$\begin{aligned}\Re\{P_4(\phi+\chi, v)\} &\geqslant (b-6v)(\phi+\Re\chi)^2 - 17(\Im\chi)^4 + 6v(\Im\chi)^2 - \frac{b^2}{2} + 3v^2 \\ &\overset{b=1+6v}{\geqslant} (\phi+\Re\chi)^1 - 17(\Im\chi)^4 - \frac{1}{2}(30v^2+12v+1) \\ &\overset{(201)}{\geqslant} \frac{\phi^2}{2} - (\Re\chi)^2 - 17(\Im\chi)^4 - \frac{1}{2}(30v^2+12v+1) \qquad (206)\end{aligned}$$

where

$$(\Re\chi)^2 + 17(\Im\chi)^4 \leqslant 17(\Re\chi)^4 + 17(\Im\chi)^4 + \tfrac{1}{68} \leqslant 17\,|\chi|^4 + \tfrac{1}{68} \qquad (207)$$

Therefrom it follows that

$$\Re\{P_4(\phi+\chi, v)\} \geqslant \frac{\phi^2}{2} - a\,|\chi|^4 - b \qquad (208)$$

with

$$a = 17, \qquad b = \tfrac{1}{2}(30v^2+12v+1) + \tfrac{1}{68} \qquad (209)$$

Consequently, we have for all $\phi \in \mathbb{R}$ and $\chi \in \mathbb{C}$ that

$$\begin{aligned}|\exp(-gP_4(\phi+\chi, v))| &= \exp(-g\Re\{P_4(\phi+\chi, v)\}) \\ &\leqslant \exp\left(-g\frac{\phi^2}{2} + ga\,|\chi|^4 + gb\right) \qquad (210)\end{aligned}$$

the elementary stability bound in ϕ^4-theory.

15.3. Real stability bound

The matter of stability is however not tied to the matter of analyticity in ϕ. Let $\phi \in \mathbb{R}$. Then we have the lower bound

$$\begin{aligned}P_4(\phi, v) = \phi^4 - 6v\phi^2 + 3v^2 &= \left(\varepsilon\phi^2 - \frac{3v}{\varepsilon}\right)^2 + (1-\varepsilon^2)\,\phi^4 + 3\left(1 - \frac{3}{\varepsilon^2}\right)v^2 \\ &\geqslant (1-\varepsilon^2)\,\phi^4 + 3\left(1-\frac{3}{\varepsilon^2}\right)v^2 \\ &\overset{\varepsilon^2=1/2}{=} \frac{\phi^4}{2} - 15v^2 \qquad (211)\end{aligned}$$

Any lower bound on $g\phi^4$ thereby implies a lower bound on $gP_4(\phi, v)$. We have the following general lower bound by a quadratic potential:

$$g\phi^4 = (\sqrt{g}\,\phi^2 - b(g))^2 + 2b(g)\sqrt{g}\,\phi^2 - b(g)^2 \geqslant 2b(g)\sqrt{g}\,\phi^2 - b(g)^2 \tag{212}$$

For the particular choice

$$b(g)\sqrt{g} = \frac{1}{\varepsilon}\,b_{QU}(g) \tag{213}$$

it reads

$$g\phi^4 \geqslant \frac{2}{\varepsilon}\,b_{QU}(g)\,\phi^2 - \frac{1}{\varepsilon^2 g}\,b_{QU}(g)^2 \tag{214}$$

Recall that $b_{QU}(g)$ indicates the g-dependent quadratic fixed point, and is given by

$$b_{QU}(g) = b_{HT}\frac{g^\rho}{1+g^\rho}, \qquad \rho = \frac{2}{4-D} \tag{215}$$

so that

$$2\rho - 1 = \frac{4}{4-D} - 1 = \frac{D}{4-D} \tag{216}$$

is positive in the range of dimensions $D \in (0, 4)$. It follows that

$$\begin{aligned}\frac{\varepsilon g}{2}\,P_4(\phi v) &\geqslant \frac{\varepsilon g}{4}\,\phi^4 - \frac{15\varepsilon g v^2}{2}\\ &\geqslant \frac{b_{QU}(g)}{2}\,\phi^2 - \frac{b_{QU}(g)^2}{4\varepsilon g} - \frac{15\varepsilon g v^2}{2}\\ &= -a_{QU}(g) + \frac{b_{QU}(g)}{2}\,\phi^2 - a_\varepsilon(g)\end{aligned} \tag{217}$$

with

$$a_\varepsilon(g) = -a_{QU}(g) + \frac{b_{QU}(g)^2}{4\varepsilon g} + \frac{15\varepsilon g v^2}{2} \tag{218}$$

For small g, the ϕ-independent terms behave powerlike:

$$a_{QU}(g) = O(g^{\rho}), \qquad \frac{b_{QU}(g)^2}{4\varepsilon g} = O(g^{2\rho-1}), \qquad \frac{15\varepsilon g v^2}{2} = O(g) \tag{219}$$

Let $\sigma = \inf\{\rho, 2\rho - 1, 1\}$. Then we have that $a_\varepsilon(g) = O(g^\sigma)$ for small g:

There exist constants C_1, σ_1, and g_1 such that for all $\phi \in \mathbb{R}$ and $g \in [0, g_1]$:

$$\exp\left(-\frac{g}{2} P_4(\phi, v)\right) \leqslant \exp(C_1 g^{\sigma_1})\, Z_{QU}(\phi, g) \tag{220}$$

Another quadratic stability bounds follows by taking $b(g)$ constant in (212).

ACKNOWLEDGMENTS

I have the pleasure to thank Andreas Pordt for his constant support and many helpful discussions, the constructive referee for pointing out a number of inaccuracies, and Frank Zielen for checking all formulas.

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